Non-smooth analysis, optimisation theory and Banach space theory

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1. Weak Asplund spaces

Let $X$ be a Banach space. We say that a function $\varphi: X \to \mathbb{R}$ is Gâteaux differentiable at $x \in X$ if there exists a continuous linear functional $x^* \in X^*$ such that
\[
x^*(y) = \lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} \quad \text{for all } y \in X.
\]

In this case, the linear functional $x^*$ is called the Gâteaux derivative of $\varphi$ at $x \in X$. If the limit above is approached uniformly with respect to all $y \in B_X$, the closed unit ball in $X$, then $\varphi$ is said to be Fréchet differentiable at $x \in X$ and $x^*$ is called the Fréchet derivative of $\varphi$ at $x$.

A Banach space $X$ is called a weak Asplund space [Gâteaux differentiability space] if each continuous convex function defined on it is Gâteaux differentiable at the points of a residual subset (i.e., a subset that contains the intersection of countably many dense open subsets of $X$) [dense subset] of its domain.

Since 1933, when S. Mazur [55] showed that every separable Banach space is weak Asplund, there has been continued interest in the study of weak Asplund spaces. For an introduction to this area, see [61] and [32]. Also see the seminal paper [1] by E. Asplund.

The main problem in this area is given next.

**Question 1.1.** Provide a geometrical characterisation for the class of weak Asplund spaces.

Note that there is a geometrical dual characterisation for the class of Gâteaux differentiability spaces, see [67, §6]. However, it has recently been shown that there are Gâteaux differentiability spaces that are not weak Asplund [58]. Hence the dual characterisation for Gâteaux differentiability spaces cannot serve as a dual characterisation for the class of weak Asplund spaces.

The description of the next two related problems requires some additional definitions.

Let $A \subseteq (0, 1)$ and let $K_A := [(0, 1) \times \{0\}] \cup [(\{0\} \cup A) \times \{1\}]$. If we equip this set with the order topology generated by the lexicographical (dictionary) ordering (i.e., $(s_1, s_2) \leq (t_1, t_2)$ if, and only if, either $s_1 < t_1$ or $s_1 = t_1$ and $s_2 \leq t_2$) then with this topology $K_A$ is a compact Hausdorff space [46]. In the special case of $A = (0, 1)$, $K_A$ reduces to the well-known double arrow space.

The first named author was supported by NSERC and by the Canada Research Chair Program. The second named author was supported by a Marsden Fund research grant, UOA0422, administered by the Royal Society of New Zealand.
Question 1.2. Is \((C(K_A), \|\cdot\|_\infty)\) weak Asplund whenever \(A\) is perfectly meagre?

Recall that a subset \(A \subseteq \mathbb{R}\) is called perfectly meagre if for every perfect subset \(P\) of \(\mathbb{R}\) the intersection \(A \cap P\) is meagre (i.e., first category) in \(P\). An affirmative answer to this question will provide an example (in ZFC) of a weak Asplund space whose dual space is not weak\(^*\) fragmentable, see [58] for more information on this problem. For example, it is shown in [58] that if \(A\) is perfectly meagre then \((C(K_A), \|\cdot\|_\infty)\) is almost weak Asplund i.e., every continuous convex function defined on \((C(K_A), \|\cdot\|_\infty)\) is Gâteaux differentiable at the points of an everywhere second category subset of \((C(K_A), \|\cdot\|_\infty)\). Moreover, it is also shown in [58] that if \((C(K_A), \|\cdot\|_\infty)\) is weak Asplund then \(A\) is obliged to be perfectly meagre.

Our last question on this topic is the following well-known problem.

Question 1.3. Is \((C(K(0,1)), \|\cdot\|_\infty)\) a Gâteaux differentiability space?

The significance of this problem emanates from the fact that \((C(K(0,1)), \|\cdot\|_\infty)\) is not a weak Asplund space as the norm \(\|\cdot\|_\infty\) is only Gâteaux differentiable at the points of a first category subset of \((C(K(0,1)), \|\cdot\|_\infty)\), [32]. Hence a positive solution to this problem will provide another, perhaps more natural, example of a Gâteaux differentiability space that is not weak Asplund.

2. The Bishop–Phelps problem

For a Banach space \((X, \|\cdot\|)\), with closed unit ball \(B_X\), the Bishop–Phelps set is the set of all linear functionals in the dual \(X^*\) that attain their maximum value over \(B_X\); that is, the set \(\{x^* \in X^*: x^*(x) = \|x^*\| \text{ for some } x \in B_X\}\). The Bishop–Phelps Theorem [4] says that the Bishop–Phelps set is always dense in \(X^*\).

Question 2.1. Suppose that \((X, \|\cdot\|)\) is a Banach space. If the Bishop–Phelps set is a residual subset of \(X^*\) is the dual norm necessarily Fréchet differentiable on a dense subset of \(X^*\)?

The answer to this problem is known to be positive in the following cases:

(i) if \(X^*\) is weak Asplund, [36, Corollary 1.6(i)];
(ii) if \(X\) admits an equivalent weakly mid-point locally uniformly rotund norm and the weak topology on \(X\) is \(\sigma\)-fragmented by the norm, [59, Theorem 3.3 and Theorem 4.4];
(iii) if the weak topology on \(X\) is Lindelöf, [49].

The assumptions in (ii) can be slightly weakened, see [37, Theorem 2]. It is also known that each equivalent dual norm on \(X^*\) is Fréchet differentiable on a dense subset on \(X^*\) whenever the Bishop–Phelps set of each equivalent norm on \(X\) is residual in \(X^*\), [57, Theorem 4.4]. Note that in this case \(X\) has the Radon–Nikodym property.

For an historical introduction to this problem and its relationship to local uniformly rotund renorming theory, see [48].

Next, we give an important special case of the previous question.
Question 2.2. If the Bishop–Phelps set of an equivalent norm \( \|\cdot\| \) defined on \((\ell^\infty(\mathbb{N}), \|\cdot\|_\infty)\) is residual, is the corresponding closed unit ball dentable?

Recall that a nonempty bounded subset \( A \) of a normed linear space \( X \) is dentable if for every \( \varepsilon > 0 \) there exists a \( x^* \in X^* \setminus \{0\} \) and a \( \delta > 0 \) such that
\[
\|\cdot\| - \text{diam}\{a \in A : x^*(a) > \sup_{x \in A} x^*(x) - \delta\} < \varepsilon.
\]

It is well-known that if the dual norm has a point of Fréchet differentiability then \( B_X \) is dentable [75].

3. The complex Bishop–Phelps property

For \( S \) a subset of a (real or complex) Banach space \( X \), we may recast the notion of support functional as follows: a nonzero functional \( \varphi \in X^* \) is a support functional for \( S \) and a point \( x \in S \) is a support point of \( S \) if \( |\varphi(x)| = \sup_{y \in S} |\varphi(y)| \).

Let us say a set is supportless if there is no such \( \varphi \).

As Phelps observed in [66] while the Bishop–Phelps construction resolved Klee’s question [51, p. 98] of the existence of support points in real Banach space, it remained open in the complex case. Lomonosov, in [52], gives the first example of a closed convex bounded convex set in a complex Banach space with no support functionals.

Question 3.1. Characterise (necessarily complex) Banach spaces which admit supportless sets.

It is known that they must fail to have the Radon–Nikodým property [52, 53].

A Banach space \( X \) has the attainable approximation property (AAP) if the set of support functionals for any closed bounded convex subset \( W \subseteq X \) is norm dense in \( X^* \). In [53] Lomonosov shows that if a uniform dual algebra \( R \) of operators on a Hilbert space has the AAP then \( R \) is self-adjoint.

Question 3.2. Characterise complex Banach spaces with the AAP. In particular do they include \( L^1[0, 1] \)?

4. Biorthogonal sequences and support points

Uncountable biorthogonal systems provide the easiest way to produce sets with prescribed support properties.

4.1. Constructible convex sets and biorthogonal families. A closed convex set is constructible [10] if it is expressible as the countable intersection of closed half-spaces. Clearly every closed convex subset of a separable space is constructible. More generally:

Theorem 4.1 ([10]). Let \( X \) be a Banach space, then the following are equivalent.

(i) There is an uncountable family \( \{x_\alpha\} \subseteq X \) such that \( x_\alpha \not\in \text{conv}(\{x_\beta : \beta \neq \alpha\}) \) for all \( \alpha \).

(ii) There is a closed convex subset in \( X \) that is not constructible.

(iii) There is an equivalent norm on \( X \) whose unit ball is not constructible.
(iv) There is a bounded uncountable system \( \{x_\alpha, \phi_\alpha\} \subseteq X \times X^* \) such that 
\( \phi_\alpha(x_\alpha) = 1 \) and \( |\phi_\alpha(x_\beta)| \leq a \) for some \( a < 1 \) and all \( \alpha \neq \beta \).

**Example** ([10]). The sequence space \( c_0 \) considered as a subspace of \( \ell_\infty \) is not constructible. Consequently, no bounded set with nonempty interior relative to \( c_0 \) is constructible as a subset of \( \ell_\infty \). In particular the unit ball of \( c_0 \) is not constructible when viewed as a subset of \( \ell_\infty \).

In particular, if \( X \) admits an uncountable biorthogonal system then it admits a nonconstructible convex set. Under additional set-theoretic axioms, there are nonseparable Banach spaces in which all closed convex sets are constructible. These are known to include: (i) the \( C(K) \) space of Kunen constructed under the Continuum Hypothesis (\( CH \)) [64], and (ii) the space of Shelah constructed under the diamond principle [73]. In consequence, these nonseparable spaces of Kunen and Shelah have the property that for each equivalent norm, the dual unit ball is weak*-separable, [10].

**Question 4.1.** Can one construct an example of a nonseparable space whose dual ball is weak*-separable for each equivalent norm using only ZFC?

In contrast, it is shown in [10] that there are general conditions under which nonseparable spaces are known to have uncountable biorthogonal systems. Suppose \( X \) is a nonseparable Banach space such that

(i) \( X \) is a dual space, or

(ii) \( X = C(K) \), for \( K \) compact Hausdorff, and one assumes Martin’s Axiom along with the negation of the Continuum Hypothesis (\( MA + \neg CH \)).

Then \( X \) admits an uncountable biorthogonal system. Part (ii) is a deep recent result of S. Todorcevic, see for example [41, p. 5].

**Question 4.2.** When, axiomatically, does a continuous function space always admit an uncountable biorthogonal system?

**4.2. Support sets.** In a related light, consider the question:

**Question 4.3.** Does every nonseparable \( C(K) \) contain a closed convex set entirely composed of support points (the tangent cone is never linear)?

In [9] it is shown that this is equivalent to \( C(K) \) admitting an uncountable semi-biorthogonal system, i.e., a system \( \{x_\alpha, f_\alpha\}_{1 \leq \alpha < \omega_1} \subseteq X \times X^* \) such that 
\( f_\alpha(x_\beta) = 0 \) if \( \beta < \alpha \), \( f_\alpha(x_\alpha) = 1 \) and \( f_\alpha(x_\beta) \geq 0 \) if \( \beta > \alpha \). Moreover, [9] observes that Kunen’s space is an example where this happens without there being an uncountable biorthogonal system assuming \( CH \). Thus, the answer is ‘yes’ except perhaps when \( MA \) fails (along with \( CH \)).

**4.3. Supportless sets.** For a set \( C \) in a normed space \( X \), \( x \in C \) is a weakly supported point of \( C \) if there is a linear functional \( f \) such that the restriction of \( f \) to \( C \) is continuous and nonzero. Fonf [35], extending work of Klee [50] (see also Borwein–Tingley [8]) proves the following result which is in striking contrast to the Bishop–Phelps theorem in Banach spaces: **Every incomplete separable normed**
space $X$ contains a closed bounded convex set $C$ such that the closed linear span of $C$ is all of $X$ and $C$ contains no weakly supported points.

Let us call such a closed bounded convex set supportless. It is known that there are Fréchet spaces (complete metrizable locally convex spaces) which admit supportless sets. In [65] Peck shows that for any sequence of nonreflexive Banach spaces $\{X_i\}$, in the product space $E = \prod_{i=1}^{\infty} X_i$, there is a closed bounded convex set that has no $E^*$-support points. Peck also provides some positive results.

**Question 4.4.** Characterise when a Fréchet space contains a closed convex supportless convex set?

5. Best approximation

Even in Hilbert spaces and reflexive Banach spaces some surprising questions remain open.

**Question 5.1.** Is there a nonconvex subset $A$ of a Hilbert space $H$ with the property that every point in $H \setminus A$ has a unique nearest point?

Such a set is called a Chebyshev set and must be closed and bounded. For a good up-to-date general discussion of best approximation in Hilbert space we refer to [27]. Asplund [2] shows that if nonconvex Chebysev sets exist then among them are so called Asplund caverns—complements of open convex bodies. In finite dimensions, the Motzkin–Klee theorem establishes that all Chebyshev sets are convex. Four distinct proofs are given in [7, §9.2] which highlight the various obstacles in infinite dimensions.

**Question 5.2.** Is there a closed nonempty subset $A$ of a reflexive Banach space $X$ with the property that no point outside $A$ admits a best approximation in $A$? Is this possible in an equivalent renorm of a Hilbert space?

The Lau–Konjagin Theorem (see [5]) states that in a reflexive space, for every closed set $A$ there is a dense (or generic) set in $X \setminus A$ which admits best approximations if and only if the norm has the Kadec–Klee property. Thus, any counter example must have a non-Kadec–Klee norm and must be unbounded—via the Radon–Nikodým property. In [5], a class of reflexive non-Kadec–Klee norms is exhibited for which some nearest points always exist.

By contrast, in every nonreflexive space, James’ Theorem [34] provides a closed hyperplane $H$ with no best approximation: equivalently $H + B_X$ is open. More exactly, two closed bounded convex sets with nonempty interior are called companion bodies and antiproximinal if their sum is open. Such research initiates with Edelstein and Thompson [31].

**Question 5.3.** Characterise Banach spaces (over $\mathbb{R}$) that admit companion bodies.

Such spaces include $c_0$ [6, 22, 31] and again do not include any space with the Radon–Nikodým property [5].
6. Metrizability of compact convex sets

One facet of the study of compact convex subsets of locally convex spaces is the determination of their metrizability in terms of topological properties of their extreme points. For example, a compact convex subset $K$ of a Hausdorff locally convex space $X$ is metrizable if, and only if, the extreme points of $K$ (denoted $\text{Ext}(K)$) are Polish (i.e., homeomorphic to a complete separable metric space), [23].

Since 1970 there have been many papers on this topic (e.g., [23, 24, 45, 54, 69] to name but a few).

Question 6.1. Let $K$ be a nonempty compact convex subset of a Hausdorff locally convex space (over $\mathbb{R}$). Is $K$ metrizable if, and only if, $A(K)$, the continuous real-valued affine mappings defined on $K$, is separable with respect to the topology of pointwise convergence on $\text{Ext}(K)$?

The answer to this problem is known to be positive in the following cases:

(i) if $\text{Ext}(K)$ is Lindelöf, [60];
(ii) if $\overline{\text{Ext}(K)} \setminus \text{Ext}(K)$ is countable, [60].

Question 6.1 may be thought of as a generalisation of the fact that a compact Hausdorff space $K$ is metrizable if, and only if, $C_p(K)$ is separable. Here $C_p(K)$ denotes the space of continuous real-valued functions defined on $K$ endowed with the topology of pointwise convergence on $K$.

7. The boundary problem

Let $(X, \|\cdot\|)$ be a Banach space. A subset $B$ of the dual unit ball $B_{X^*}$ is called a boundary if for any $x \in X$, there is $x^* \in B$ such that $x^*(x) = \|x\|$. A simple example of boundary is provided by the set $\text{Ext}(B_{X^*})$ of extreme points of $B_{X^*}$. This notion came into light after James’ characterisation of weak compactness [44], and has been studied in several papers (e.g., [16–19, 38–40, 70, 74, 76]). In spite of significant efforts, the following question is still open (see [38, Question V.2] and [30, Problem I.2]):

Question 7.1. Let $A$ be a norm bounded and $\tau_p(B)$ compact subset of $X$. Is $A$ weakly compact?

The answer to the boundary problem is known to be positive in the following cases:

(i) if $A$ is convex, [74];
(ii) if $B = \text{Ext}(B_{X^*})$, [12];
(iii) if $X$ does not contain an isomorphic copy of $\ell_1(\Gamma)$ with $|\Gamma| = c$, [17, 18];
(iv) if $X = C(K)$ equipped with their natural norm $\|\cdot\|_\infty$, where $K$ is an arbitrary compact space, [16].

Case (i) can be also obtained from James’ characterisation of weak compactness, see [39]. The original proof for (ii) given in [12] uses, among other things, deep results established in [11]. Case (iii) is reduced to case (i): if $\ell_1(\Gamma) \not\subset X$,
The existence problem, i.e., if \( f \) is separately continuous find conditions on either \( X \) or \( Y \) such that \( f \) is jointly continuous at each \((x, y) \in X \times Y\). Let \( \Gamma \subset X \times Y \) be a Baire space and let \( \Gamma \) be a compact Hausdorff space. There have been many contributions to this area (e.g., [16, 18]).

We observe that the solution in full generality to the boundary problem without the concourse of James' theorem of weak compactness would imply an alternative version of James' theorem itself: a Banach space \( X \) is reflexive if, and only if, each element \( x^* \in X^* \) attains its maximum in \( B_X \).

Finally, we point out that in the papers [71, 79], it has been claimed that the boundary problem was solved in full generality. Unfortunately, to the best of our knowledge both proofs appear to be flawed.

8. Separate and joint continuity

If \( X, Y \) and \( Z \) are topological spaces and \( f: X \times Y \rightarrow Z \) is a function then we say that \( f \) is jointly continuous at \((x_0, y_0) \in X \times Y \) if for each neighbourhood \( W \) of \( f(x_0, y_0) \) there exists a product of open sets \( U \times V \subseteq X \times Y \) containing \((x_0, y_0)\) such that \( f(U \times V) \subseteq W \) and we say that \( f \) is separately continuous on \( X \times Y \) if for each \( x_0 \in X \) and \( y_0 \in Y \) the functions \( y \mapsto f(x_0, y) \) and \( x \mapsto f(x, y_0) \) are both continuous on \( Y \) and \( X \) respectively.

Since the paper [3] of Baire first appeared there has been continued interest in the question of when a separately continuous function defined on a product of nice spaces admit a point (or many points) of joint continuity and over the years there have been many contributions to this area (e.g., [15, 20, 21, 25, 26, 49, 56, 63, 68, 72, 77] etc.). Most of these results can be classified into one of two types. (I) The existence problem, i.e., if \( f: X \times Y \rightarrow \mathbb{R} \) is separately continuous find conditions on either \( X \) or \( Y \) (or both) such that \( f \) has at least one point of joint continuity. (II) The fibre problem, i.e., if \( f: X \times Y \rightarrow \mathbb{R} \) is separately continuous find conditions on either \( X \) or \( Y \) (or both) such that there exists a nonempty subset \( R \) of \( X \) such that \( f \) is jointly continuous at the points of \( R \times Y \).

The main existence problem is, [78]:

**Question 8.1.** Let \( X \) be a Baire space and let \( Y \) be a compact Hausdorff space. If \( f: X \times Y \rightarrow \mathbb{R} \) is separately continuous does \( f \) have at least one point of joint continuity?

We will say that a topological space \( X \) has the Namioka Property if for every compact Hausdorff space \( Y \) and every separately continuous function \( f: X \times Y \rightarrow \mathbb{R} \) there exists a dense \( G_\delta \)-subset \( G \) of \( X \) such that \( f \) is jointly continuous at each
point of $G \times Y$. Similarly, we will say that a compact Hausdorff space $Y$ has the co-Namioka Property if for every Baire space $X$ and every separately continuous function $f: X \times Y \to \mathbb{R}$ there exists a dense $G_\delta$-subset $G$ of $X$ such that $f$ is jointly continuous at each point of $G \times Y$.

The main fibre problems are:

**Question 8.2.** Characterise the class of Namioka spaces.

There are many partial results.

(i) Every Namioka space is Baire, \([72]\);

(ii) Every separable Baire space and every Baire $p$-space is a Namioka space, \([72]\);

(iii) Not every Baire space is a Namioka space, \([78]\);

(iv) Every Lindelöf weakly $\alpha$-favourable space is a Namioka space, \([49]\);

(v) Every space expressible as a product of hereditarily Baire metric spaces is a Namioka space, \([20]\).

**Question 8.3.** Characterise the class of co-Namioka spaces.

There are many partial results.

(i) $\beta\mathbb{N}$ is not a co-Namioka space, \([28]\);

(ii) Every Valdivia compact is a co-Namioka space, \([13, 29]\);

(iii) The co-Namioka spaces are stable under products, \([15]\);

(iv) All scattered compacts $K$ with $K^{(\omega_1)} = \emptyset$ are co-Namioka, where $K^{(\alpha)}$ denotes the $\alpha$th derived set of $K$, \([28]\);

(v) There exists a non co-Namioka compact space $K$ such that $K^{(\omega_1)}$ is a singleton, \([43]\).

A partial characterisation, in terms of a topological game on $C_p(K)$, is given in \([47]\) for the class of compact spaces $K$ such that: for every weakly $\alpha$-favourable space $X$ and every separately continuous mapping $f: X \times K \to \mathbb{R}$ there exists a dense $G_\delta$ subset $G$ of $X$ such that $f$ is jointly continuous at each point of $G \times K$.

For an introduction to this topic, see \([56, 68]\). Also see the seminal paper \([62]\) by I. Namioka, as well as, the paper \([63]\).

Acknowledgements

The material on the Boundary problem (§ 7) was provided by B. Cascales.

References

References


Topological structures of ordinary differential equations

V. V. Filippov

Basic elements of the topological structures in questions look as follows.

Let $U$ be a subset of the product $\mathbb{R} \times \mathbb{R}^n$. When considering differential equations $y' = f(t, y)$ or inclusions $y' \in f(t, y)$ having a right-hand side defined on the set $U$, we call a function $z$ a solution to the equation if the graph of the function $z$ lies in $U$, and (letting $\text{dom}(z)$ denote the domain of the function $z$)

(a) $\text{dom}(z)$ is a segment, $z$ is generalized absolutely continuous and $z'(t) = f(t, z(t))$ (respectively, $z'(t) \in f(t, z(t))$) for almost all $t \in \text{dom}(z)$, or

(b) $\text{dom}(z)$ a singleton.

For equations with a continuous right-hand side our definition gives only continuously derivable solutions. For equations with a right-hand side satisfying the Caratheodory conditions this definition gives Caratheodory solutions.

In this definition we do not fix domains of functions. We consider functions with various domains together. We take as a distance between two solutions the Hausdorff distance between their graphs.

Let us emphasize some basic properties of the set $Z$ of so defined solutions.

(1) If $z \in Z$ and a segment $I$ lies in $\text{dom}(z)$ then $z|_I \in Z$.

(2) If $z_1, z_2 \in Z$, $I = \text{dom}(z_1) \cap \text{dom}(z) \neq \emptyset$ and $z_1|_I = z_2|_I$ then the function

$$z(t) = \begin{cases} z_1(t) & \text{if } t \in \text{dom}(z_1) \\ z_2(t) & \text{if } t \in \text{dom}(z_2) \end{cases}$$

(defined on the segment $\text{dom}(z_1) \cup \text{dom}(z_2)$) belongs to the set $Z$.

(c) The set $Z_K = \{ z \in Z : \text{the graph of } z \text{ is a subset of } K \}$ is compact for every compact $K$ subset of $U$.

(e) For each point $(t, y)$ of the set $U$ there exists a function $z \in Z$ such that $t$ is in the interior of $\text{dom}(z)$ and $z(t) = y$.

(u) If $z_1, z_2 \in Z$, $\text{dom}(z_1) = \text{dom}(z_2)$ and $z_1(t) = z_2(t)$ for some $t \in \text{dom}(z_1)$ then $z_1 = z_2$.

It is easy to see that the properties (1) and (2) follow directly from our definitions. Conditions (e) and (u) correspond to the existence theorem and the uniqueness theorem. Condition (c) corresponds to (upper semi-)continuity of the dependence of solutions to Cauchy problems on initial values.

The property of solution sets which corresponds to the continuity of the dependence of solutions on parameters looks as follows. $R(U)$ denotes the set of all sets of functions which are defined on segments and singletons and whose graphs lie in $U$, satisfying conditions (1) and (2).
$\{1, 2, \ldots\} \subseteq R(U)$ converges (in $U$) to a space $Z \in R(U)$ if for any compact $K$ subset of $U$ and for any sequence $z_i \in (Z_{j_i})_K$ ($j_1 < j_2 < \cdots$) there is a subsequence $\{z_{i_m} : m = 1, 2, \ldots\}$ converging to a function $z \in Z$.

The notation $R_*(U)$ denotes the set of all elements of $R(U)$ satisfying all conditions listed in the subscript ($*$ may include c, e or u as above). The sets $R_*(U)$ are called classes of solution spaces or spaces of solution spaces.

On the class $R_c(U)$ the introduced convergence corresponds to a non-$T_1$ first countable topology.

It is surprising that these simple conditions suffice to account for a considerable part of the theory of ordinary differential equations, replacing the usual conditions of the continuity of the right-hand side. See [2] for details.

The first topological ideas arose in Analysis and Geometry. They passed to general topological notions and constructions when mathematicians felt that topological relations appeared more often in Mathematics that was previously understood. This implied many important consequences. It may be that the most brilliant consequence between those subjects was the creation of Functional Analysis.

But when (General) Topology was created it had many internal reasons for its development. The initial motivation, related to the necessity to serve other domains of Mathematics, was largely forgotten. The feeling of this omission encouraged me to try to compare the contents of General Topology itself and the contents of other domains of Mathematics.

Perhaps the most interesting observation in this direction was made when I saw that the analysis of the continuity of the dependence of solutions of ordinary differential equations on initial values and parameters leads to a topological structure of S. Nedev’s type[4]. It was the space of solution spaces $R_{ce}(U)$. This first interest was related to the observation that Nedev’s results give real information about the structure of this space. In particular, Nedev’s theorems imply the metrizability of some subspaces of $R_{ce}(U)$. Later I understood that this topological structure gives a powerful tool for the theory of ordinary differential equations itself. Notions such as first approximation, asymptotically autonomous spaces receive here their natural importance without loss of possibility of their application. See below.

The new topological structure allows us to develop efficiently a theory which deals easily with equations having singularities and with equations with multivalued right-hand sides (differential inclusions). It extends the majority of assertions of the central part of the theory of ordinary differential equations in the existing framework to equations with complicated discontinuities of right-hand sides. The simplest example on this direction looks as follows.

**Example.** Let the functions $f, g : \mathbb{R} \to [1, 2]$ be measurable. The equation

$$y' = f(t) + g(y)$$

is far from the classical theory because it may have discontinuities both in time and in space variable but it is covered well by our approach.
Our approach reinforces the theory in the case of equations with continuous right-hand sides too.

Here are some other consequences. The notion of a dynamical system and that of an autonomous solution space, in which a solution for every Cauchy problem exists, is unique and is defined on the whole real line, give a different axiomatic description of the same object. So our results may be applied to studies of dynamical systems too.

So this research of a non-traditional topological description of relations in a domain of Mathematics was successful. A general problem arises.

**Problem 1.** Find other as yet undiscovered topological relations in Mathematics and to try to use them to reinforce existing mathematical theories.

This invitation contains nothing new. Such a rôle of topological structures (so, of General Topology) in Mathematics was highly praised in the famous N. Bourbaki’s article *The Architecture of Mathematics* [1]. So the problem is to show that Bourbaki’s appreciation is not exhausted by known cases of applications of topological structures.

In particular, some parts of the theory of partial differential equations and of equations in Banach spaces are close to the theory of ordinary differential equations. So, I ask:

**Problem 2.** For which problems of the theory of partial differential equations and of equations in Banach spaces can this method be applied?

One of first consequences of the usage of new structures for ordinary differential equations was a method of investigation of singularities. In the domain $U$ under consideration we find open subsets $V, V \in \gamma$, which do not contains singularities. Then we try to find estimates of the remainder $R = U \setminus \bigcup \gamma$ which show that singularities lying in $R$ do not influence the properties of solutions. The level of new topological structures is very suitable for this consideration. The simplest example on this direction looks as follows.

**Example.** Let $f(t, y)$ be a polynomial. Then solutions to the equation

$$y' = f(t, y) + \frac{\alpha}{t^2 + y^2 + \alpha^2}$$

depend continuously on the parameter $\alpha$, although for $t = y = 0$ the second term tends to infinity when $\alpha \to 0$.

**Problem 3.** For which other problems of the theory of ordinary differential equations (and outside it) can this method of study of singularities (using partial mappings) be applied?

When we investigate a particular equation with singularities we need to prove the fulfillment of the above-listed basic conditions. We get this purpose in whole measure if we prove that the solution space $Z$ in question belongs to the closure of the class $R_{ceu}(U)$: $Z \in \text{cl}_{R_{ceu}(U)} R_{ceu}(U)$. Really, the condition $Z \in \text{cl}_{R_{ceu}(U)} R_{ceu}(U)$ assures that the equation under consideration is covered by the theory completely. The following question remains unanswered:
Problem 4. Suppose that the domain $U$ is covered by a family $\gamma$ of open subsets and $Z_V \in \text{cl}_{R_{ce}(U)} R_{ce}(V)$ for every $V \in \gamma$. Is necessarily $Z \in \text{cl}_{R_{ce}(U)} R_{ce}(U)$?

Now many chapters of the theory of ordinary differential equations are covered by the axiomatic approach. But the theory is very large and the problem to cover the entire theory will remain current for a long time. I do not think that all the topological effects of the theory of ordinary differential equations have been discovered yet.

In this investigation the main problems are not technical. The main problems are to understand topological contents of corresponding notions, constructions, and theorems.

Example (First approximation). Usually they consider a vector equation

$$y' = Ay + g(y),$$

where $A$ is a matrix and $g(y)$ is small with respect to $\|y\|$: $\|g(y)\| = o(\|y\|)$. Instead, we consider the family of changes of variables, corresponding to homotheties $y \to \lambda y$, where $\lambda > 0$. This change of variables transforms our equation to the equation

$$y' = f(y) + \lambda g\left(\frac{1}{\lambda} y\right),$$

Denote its solution space by $Z_\lambda$. In our approach we replace the usual condition of first approximation by the convergence of $Z_\lambda$ to the solution space of the equation

$$y' = Ay$$

as $\lambda \to \infty$ in the topological space $R_{ce}(U)$. We return to the classical version of this notion when we prove this convergence using the classical theorem on continuous dependence of solution on parameter $\lambda$.

Example (Asymptotically autonomous space). Usually they consider a vector equation

$$y' = f(y) + g(t, y),$$

where the term $g(t, y)$ has an estimate $\|g(t, y)\| \leq \phi(t)$ where the function $\phi$ is small at infinity. This means that $\phi(t) \to 0$ when $t \to \infty$ or that the function $\phi$ is integrable on $(a, \infty)$. Instead, we consider the family of changes of variables, corresponding to translation along the time axis. Such translation transforms our equation to equation

$$y' = f(y) + g(t + \tau, y),$$

Denote its solution space by $Z_\tau$. In our approach we require the convergence of $Z_\tau$ to the solution space of the equation

$$y' = f(y),$$

as $\tau \to \infty$ in the topological space $R_{ce}(U)$.

Because methods of proof of convergence in the space $R_{ce}(U)$ are largely developed, there is a larger volume of new versions of notions of first approximation and of asymptotically autonomous spaces. Each theorem of the classical theory where
those notions are used obtains a broader scope (and a possibility of application to equations and inclusions with discontinuous right-hand sides).

One of the consequences of this study was the creation of a new version of the translation method in the theory of boundary value problems in which the continuation principle works. Before, in the corresponding situation they used the Leray–Schauder theory. But the Leray–Schauder theory in general does not work with equations with a discontinuous right-hand side. Our new approach works; see [3] and my other articles. The new version of the translation method is based on studies of Čech homology of solution spaces. Perhaps something similar may be made by investigating shape properties of solution spaces. So, I ask:

**Problem 5.** Is it possible to create an approach to the theory of boundary value problems based on shape properties of solution spaces and on essential mappings? 
(Of course, without assuming the uniqueness of solutions of the Cauchy problem.)

It may be simpler than the application of Čech homology.

**References**


The interplay between compact spaces and the Banach spaces of their continuous functions

Piotr Koszmider

Introduction

We will consider compact (always Hausdorff and infinite) spaces K and the set C(K) of all continuous functions from K into the reals. If C(K) is equipped with the supremum norm, it is a Banach space. An isomorphism between Banach spaces is a linear isomorphism which is continuous (necessarily both ways, by the open mapping theorem). A C(K) will mean a Banach space of the form C(K) for some compact K. As general references that might be useful while reading this article we suggest [9, 16, 49] on functional analysis, [22, 33] on set theory and [15] on topology.

If T: C(K) → C(L) is an isometry (i.e., a linear isomorphism which preserves the norm), T induces a homeomorphism between K and L (Banach–Stone, see [49, 7.8.4]), but many nonhomeomorphic Ks have isomorphic C(K)s.

The simplest examples are of two disjoint convergent sequences K, i.e., (y_{2m}) → ∞₁ and (y_{2m−1}) → ∞₂ for m > 0 with their respective distinct limit points ∞₁ and ∞₂ and one convergent sequence L, i.e., (x_n) → ∞ with its limit point ∞ (see e.g. [2] for more). One can explicitly define an isomorphism T:

\[ T(f)(x_0) = f(\infty_1) - f(\infty_2), \]
\[ T(f)(x_{2m}) = f(y_{2m}) - \frac{1}{2}(f(\infty_1) - f(\infty_2)), \]
\[ T(f)(x_{2m−1}) = f(y_{2m−1}) + \frac{1}{2}(f(\infty_1) - f(\infty_2)), \]
\[ T(f)(\infty) = \frac{1}{2}(f(\infty_1) + f(\infty_2)). \]

for m > 0. This is clearly also an example of a C(K) which is both a quotient (image under an onto linear and continuous mapping between Banach spaces) and a subspace (for us always meaning a closed linear subspace) of a C(L) such that K is neither a subspace nor a continuous image of L.

Working with C(K)s then seems to be a weaker environment than working with compact spaces, because we identify compact spaces with the same C(K) (in the isomorphic sense). It is just one point of view. From another perspective, we get more continuous mappings from K to K, i.e., not all operators on C(K) come from a usual continuous mapping on K. The problems of this article could

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1 Other structures include: Banach algebra, C∗-algebra, lattice, ring, topological vector space with various topologies.

The author was partially supported by a research fellowship Produtividade em Pesquisa from National Research Council of Brazil (Conselho Nacional de Pesquisa, Processo Número 300369/01-8).
be summed up as: **What can we say about \( C(K) \) if we know some topological properties of \( K \)?** (impose properties on \( C(K) \) by manipulating \( K \)’s); or: **If \( C(K) \) and \( C(L) \) are related in the isomorphic sense, how are \( K \) and \( L \) related in the topological sense?** (get different \( C(K) \)’s knowing that \( K \)’s are very different). This is certainly a very special and topological point of view since most of the theory of \( C(K) \) spaces is developed having in mind only Banach space theory (for example, asking similar questions about the dual ball with the weak∗ topology instead of \( K \)). However as history shows, a \( K \) with sufficiently strong topological properties can produce a striking example of a Banach space. Another bias of the article is focusing on the isomorphic structure of subspaces, quotients and complemented subspaces on the \( C(K) \)’s.

The ideal results here would be describing which \( K \)’s have isomorphic \( C(K) \)’s. There are only a few such results. For example, \( C(K) \) is isomorphic to \( C([0,1]) \) if and only if \( K \) is an uncountable metrizable compact space ([38]). Similarly one could characterize \( C(K) \)’s for countable \( K \)’s ([6]) or \( C(K) \)’s isomorphic to \( C(L) \) where \( L \) is the one point compactification of a discrete space, i.e., is isomorphic to a \( c_0(\kappa) \) for some cardinal \( \kappa \) ([36]).

Another natural type of an interesting result is to prove that if \( C(K) \) and \( C(L) \) are isomorphic and \( K \) has some topological property, then \( L \) has it as well. This holds for properties such as being dispersed, being Eberlein, being ccc, being metrizable and many others. Probably the most well-known (see [23] for definitions and related theory) open problem here is the following:

**Question 1.** If \( K \) is a Corson compact and \( C(L) \) is isomorphic to \( C(K) \), must \( L \) be a Corson compact?

This is true if we assume \( \text{MA} + \neg \text{CH} \), (see [4]).

As noted above, the Banach–Stone theorem gives a special place to isometries between \( C(K) \)’s, hence even though we are interested in the isomorphic theory, the isometries will be mentioned. We have isometries between \( \ell_\infty \) and \( C(\beta\mathbb{N}) \), \( c \) and \( C([0,\omega]) \), \( C(\omega^*) \) and \( \ell_\infty/c_0 \) (see [35]). Also \( c_0 \) and \( c \) are isomorphic (but not isometric).

The main tools of functional analysis include the use of the dual and the bidual of the Banach space. In the case of \( C(K) \)’s we are in a very privileged situation, (like in the case of \( \ell_p \) or \( L_p(\mu) \) spaces), i.e., we can see the functionals and even the elements of the bidual with the naked eye. The Riesz representation theorem (see [49, 18]) says that any continuous functional \( \phi \) on a \( C(K) \) can be isometrically associated with a unique Radon2 measure \( \mu \) on \( K \) by the formula \( \phi(f) = \int f d\mu \) for all \( f \in C(K) \) where the integration is like in the Lebesgue theory. Even the norm of \( \phi \) is nicely describable by \( \mu \); it is the variation of \( \mu \), i.e., the supremum of the expressions of the form \( |\mu(A_1)| + \cdots + |\mu(A_n)| \) for \( A_i \) pairwise disjoint and Borel. The addition of measures and multiplication by a scalar is setwise. Thus,

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2A Radon measure here is a Borel, countably additive, signed, regular measure. Regularity for a signed measure means that for any Borel \( A \subseteq K \) the value of \( |\mu|(U - F) \) is arbitrarily small for some open \( U \) and compact \( F \) such that \( F \subseteq A \subseteq U \).
we will use letters $\mu$, $\nu$, $\lambda$ for elements of the dual of $K$ (as a Banach space) which will be identified with the Banach space $M(K)$ of the Radons on $K$ with the variation norm. As usual, one can decompose the Radon measure $\mu$ into its positive and negative parts $\mu_+, \mu_-$ and define $\mu_+ - \mu_- = |\mu|$ which is a positive Radon measure (see [49, 17.2.5, 18.2.5]). $M(K)$ has a natural topology where $K$ with its original topology is a subspace, i.e., the weak$^*$ topology defined by the subbasis of the sets of the form $\{\mu \in M(K) : \mu(f) \in I\}$ where $f \in C(K)$ and $I$ is an open interval of the reals. Here $\{\delta_x : x \in K\}$ is a copy of $K$ where $\delta_x(A) = 1$ if $x \in A$ and is 0 otherwise. This copy of $K$ is quite big; its span is weak$^*$ dense in $M(K)$.

With the help of the dual one can see how the $C(K)$ partially loses information about $K$. If $T : C(K) \to C(L)$ is an operator (i.e., a linear continuous function), define $T^* : M(L) \to M(K)$ by $T^*(\nu) = \nu \circ T$. Deciphering it in terms of the integration we get $\int f d(T^*(\nu)) = \int T(f) d\nu$. For example, if $\nu$ is the simplest Radon measure, the Dirac measure $\delta_x$ concentrated on a point $x \in K$, we have $\int T(f) d\delta_x = T(f)(x)$. That is, $T(f)(x)$ is the value of the functional $T^*(\delta_x)$ on $f$. If $T$ is given by $T(f) = f \circ F$ where $F : K \to K$ is continuous, then $T(f)(x) = f(F(x))$, i.e., $T^*(\delta_x) = \delta_{F(x)}$. In other words, $T^*$ essentially is $F$. However, in general $T^*(\delta_x)$ may be some more complicated measure, and in this way it loses information about $K$. In the example from the beginning of this section we have $T^*(\delta_{x_0}) = \delta_{x_0}^\infty - \delta_{x_0}^\infty$. No continuous function from $L$ into $K$ sends a point onto a linear combination of two points. Thus, one way of proving negative properties of $C(K)$ is to strengthen the topological properties of $K$, taking care not only of point-to-point continuous functions but also taking care of point-to-measure weak$^*$ continuous functions. Note that by knowing that the span of the pointwise measures $\delta_x$ for $x \in K$ is dense in the weak$^*$ topology in $M(K)$ and that $T^*$ is always continuous in the weak$^*$ topologies, we may really restrict our attention to $T^*(\delta_x)$s to obtain complete information about $T^*$.

If points of $K$ can be considered as functionals on $C(K)$, then functions of $K$ should be functionals on functionals, i.e., elements of the bidual. Indeed $C(K)$, as any Banach space, canonically embeds in its bidual, but also any bounded Borel function $g : K \to \mathbb{R}$ defines a functional $\Psi_g$ on $M(K)$ by $\Psi_g(\mu) = \int g d\mu$. As points span a weak$^*$ dense set in $M(K)$, the borel sets (i.e., their characteristic functions) span a weak$^*$ dense set in the bidual of $C(K)$. If $X \subseteq M(K)$ is a separable subspace, then by the Radon–Nikodým theorem there is an isometry of $X$ and a subspace of $L_1(\mu)$ for some $\mu \in M(K)$ (just take $\mu = \sum |\mu_n|/2^n ||\mu_n||$ where $\{\mu_n : n \in \mathbb{N}\}$ is dense in $X$) and so $L_\infty(\mu)$ is the bidual of a superspace of this separable piece of $M(K)$ ([49, 27.1.3]). More on entire biduals of $C(K)$s can be found in [25, 26] and [49, 27.2].

This world of dualities can be even more tangible and combinatorial if we are allowed to think about $K$ as a dual of something more primitive, namely the

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3This observation may serve to note that a $C(K)$ is reflexive as a Banach space iff the characteristic function of any Borel set is continuous, i.e., if and only if $K$ is finite if and only if $C(K)$ is finite-dimensional. So, for example, $\ell_2$ is not isomorphic to any $C(K)$. 

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Boolean algebra of clopen sets, such as when $K$ is totally disconnected and Stone duality (see [28]) may enter the game. Then, by the Weierstrass–Stone theorem (see [49, 7.3]) the finite linear combinations of characteristic functions of clopen sets form a norm-dense subset of the $C(K)$, i.e., the span of the Boolean algebra is dense in $C(K)$. (Then the Hanh–Banach theorem and the Tarski ultrafilter theorem become one.) Radon measures on totally disconnected $K$ are completely determined by their restrictions to clopen sets and the dual space may be interpreted as the space of finitely additive bounded signed measures of the Boolean algebra ([49, 18.7]).

If $K$ is metrizable, then $C(K)$ is isomorphic to a $C(L)$ for $L$ totally disconnected. Indeed, we have a classification of separable $C(K)$s, i.e., those whose $K$'s are metrizable (see [49, 7.6.5]): such $C(K)$s are isomorphic to $C(2^\omega)$, $C([0, \omega])$ or $C([0, \alpha])$ for $\alpha < \omega_1$ such that $\beta^n < \alpha$ for all $\beta < \alpha$ and $n \in \omega$ (due to Miltutin [38], Bessaga, Pełczyński [6]). Here we need to admit another bias of this article: we are mainly interested in nonseparable $C(K)$s. The issues related to separable $C(K)$s are more analytic and are presented in detail in [47]. One wonders if any $C(K)$ is isomorphic to a $C(L)$ for $L$ totally disconnected; the latter type of a $C(K)$ will be called Boolean in the sequel. Only recently it turned out that it is not the case ([30, 43]), and the reason is quite ad hoc; the $C(K)$ of [30] is indecomposable. So the question remains when and why a $C(K)$ is Boolean.

**Question 2.** If $K$ is small compact space, that is, (a) first countable, (b) of weight smaller than $2^\omega$ under $\text{MA} + \neg \text{CH}$, or (c) Eberlein, is $C(K)$ isomorphic to a Boolean $C(L)$?

For more on these kinds of questions and the methods that are being used to answer them, see the section on complemented subspaces. Regardless of the results of [30] one may still hope for some theorem which would explain the special role of Boolean $C(K)$s not just in heuristic terms. For example, one could hope for some transfer principle which would imply general statements about $C(K)$s from those proved about Boolean $C(K)$s. Consider the following:

**Question 3.** Does every nonseparable $C(K)$ have a subspace which has a quotient isomorphic to a nonseparable Boolean $C(L)$?

If the answer were positive, one could obtain some uncountable objects (those which are preserved when going to superspaces and preimages of operators) in any nonseparable $C(K)$ just by knowing that they exist in those with $K$ nonmetrizable and totally disconnected\(^4\). Note that there is a dual result ([11]): every $C(K)$ is complemented (and hence both a quotient and a subspace, see the section on complemented subspaces for definitions) in a Boolean $C(K)$ of the same density. See [5] for more research on these issues.

\(^4\)For example, we knew since [52] that it is consistent that every nonseparable Boolean $C(K)$ has an uncountable biorthogonal system. However, only recently ([53]) we have a separate proof that it is consistent for any $C(K)$. 

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The above question however shouldn’t be about complemented subspaces of $C(K)$s since the indecomposable $C(K)$ of [30] would be a counterexample. There is a topological version of the previous question:

**Question 4.** Is it consistent that every nonmetrizable compact $K$ has a continuous image with a nonmetrizable compact subspace which is totally disconnected?

We mention here only the consistency since we have a counterexample under CH.

**Subspaces**

If $F : K \to L$ is a continuous onto mapping, then $T_F(f) = f \circ F$ is an isometry of $C(L)$ and a subspace of $C(K)$. That is, continuous images of topological spaces produce isometric subspaces of $C(K)$s.

The Banach–Mazur theorem ([49, 8.7]) says that any separable Banach space is isometric to a subspace of a $C([0,1])$. In particular, it is easy for a subspace of a $C(K)$ not to be isomorphic to any space of the form $C(L)$. In general, by simple duality any Banach space is a subspace of a $C(K)$ where $K$ is its dual ball with the weak* topology. Just send $x$ to $x^{**}$ in the bidual and restrict it to the dual ball. The weight of this $K$ in the weak* topology is the density of $X$. Thus every Banach space embeds isometrically in a $C(K)$ of the same density. Under CH, every compact $K$ of weight $\leq 2^\omega$ is a continuous image of $\omega^*$. Thus, we have that under CH any Banach space of density not bigger than $2^\omega$ is a subspace of $C(\omega^*) \equiv \ell_\infty/c_0$. It is well known that without CH spaces like $[0, \omega_2]$ are not continuous images of $\omega^*$. What about the $C(K)$ analog?

**Question 5.** (a) Is it consistent that $C(\omega^*) \equiv \ell_\infty/c_0$ does not contain an isomorphic copy of some Banach space of density not bigger than $2^\omega$? (b) Can this space be $C([0,\omega_2])$?

Of course one may state the above question for other spaces instead of $\omega^*$, for example, spaces which consistently are $2^\omega$-Parovičenko (see [12]), or in general:

**Question 6.** Is it provable in ZFC that there is a compact $K$ of weight $\leq 2^\omega$ such that every Banach space of density $\leq 2^\omega$ is isomorphic to a subspace of $C(K)$?

This is related to many very deep and influential on Banach space theory successful attempts of characterizing the existence of copies of some Banach space inside a $C(K)$ space in terms of topological properties of $K$. For example, for many infinite cardinals $\kappa$ it is consistent that $C(K)$ has a subspace isomorphic to $\ell_1(\kappa)$ if and only if $K$ maps onto $[0,1]^\kappa$. An excellent survey of this gigantic project developed over a few decades by Argyros, Fremlin, Haydon, Pelczyński, Plebanek and others is [42]. There one can find many related references and open questions. Another simple example of this kind of inquiry could be that an isomorphic copy of $c_0(\omega_1)$ is a subspace of $C(K)$ if and only if $K$ is not ccc: $c_0(\omega_1)$ has a weakly compact subset which is not separable, thus by a result of [46], which says that $K$ is ccc if and only if weakly compact subsets of $C(K)$ are separable, $K$ is not
However one can easily check that there is no compact $L$ for which it is true that for every compact $K$ the space $L$ is a continuous image of $K$ if and only if $K$ is not ccc.

**Quotients**

If $K$ is a subspace of $L$ then $T_K : C(L) \to C(K)$ given by $T(f) = f|_K$ is an onto operator, i.e., topological subspaces produce quotients in function spaces, $C(L)/\text{Ker}(T)$ is isometric to $C(K)$. Our example of one and two convergent sequences shows that there may be more quotients of a $C(K)$ than subspaces of $K$. Also spaces like $\ell_2$, i.e., not isomorphic to $C(K)$s, may be quotients of $C(K)$s (see [34]). One should remember that the image of a linear operator defined on a Banach space does not have to be a quotient, as the image of a linear continuous bijection does not have to be an isomorphism. Simply, the images may not be Banach spaces; they may not be complete.

Two basic examples of compact spaces are a convergent sequence $[0, \omega]$ and $\beta\mathbb{N}$, the Čech–Stone compactification of the integers. Since for a compact space having a copy of $\beta\mathbb{N}$ as a subspace is equivalent to having $[0, 1]^{2^{\omega}}$ as a continuous image, and for a Banach space having $\ell_\infty$ as a quotient is equivalent to having $\ell_1(2^{\omega})$ as a subspace, the question of $\ell_\infty$ as a quotient is more related to the fragment of the previous section where we referred the reader to [42]. Thus we will concentrate on $c_0$ as a quotient.

For a $C(K)$ not having $c_0$ as a quotient is equivalent to a well-known Banach space theory property of Grothendieck (see [48, 5.1. ii) and 5.3]). A Banach space $X$ is said to have the Grothendieck property if weak∗ convergent sequences in the dual $X^*$ are weakly convergent. This is what we get in the Banach space language about $X = C(K)$ if we want to guarantee that $K$ has no convergent sequences. Indeed, this roughly says that if a bounded sequence of measures, like for example $(\delta_{x_n})_{n \in \mathbb{N}}$, is separated into two parts in the weak topology on $M(K)$, e.g., by a Borel subset, then it is separated by a continuous function on $K$. In particular if a $C(K)$ has the Grothendieck property then $K$ has no nontrivial convergent sequences. The Grothendieck property is stronger. However no perfect solution exists, i.e., having a convergent sequence is not a Banach space theory property. Consider $K$, the Stone space of the subalgebra of $\mathcal{P}(\mathbb{N})$ of all subsets $a$ of $\mathbb{N}$ such that $2n \in a$ if and only if $2n+1 \in a$ for all but finitely many $n \in \mathbb{N}$. It is easy to see that, like $\beta\mathbb{N}$, it has no convergent sequences, but $P(f) = (f(2n) - f(2n - 1))_{n \in \mathbb{N}}$ defines an operator from $C(K)$ onto $c_0$ ([48, 4.10]). Also $C(K) \sim \ell_\infty \oplus c_0 \sim C(L)$ where $L$ is a disjoint union of $\beta\mathbb{N}$ and $[0, \omega]$. That is, $C(K)$s can be isomorphic even though one $K$ has convergent sequences and the other does not. One should also realize that a $C(K)$ without the Grothendieck property may verify it for all atomic measures; there are even $C(K)$s without the Grothendieck property such that for every separable $L \subseteq K$, the space $C(L)$ has the Grothendieck property ([44]). Since [48] the following question attributed to Lindenstrauss is left open:

**Question 7.** Can we characterize topologically compact $K$’s such that $C(K)$ has the Grothendieck property?
There is another Banach space theoretic nonequivalent way of guaranteeing that a totally disconnected $K$ has no convergent sequences: require that $C(K)$ has the Nikodým property, i.e., whenever elements $\mu_n$ of $M(K)$ have their values $\mu_n(a)$ bounded for each clopen $a \subseteq K$, then $\mu_n$ are all norm-bounded ([48, 4.6]). If $K$ is the Stone space of the Boolean algebra of Jordan measurable subsets of $[0,1]$ then $K$ has the Nikodým property but does not have the Grothendieck property ([48, 3.2., 3.3]). Assuming CH Talagrand constructed a Boolean algebra whose Stone space has the Grothendieck property but lacks the Nikodým property ([51]). Here the main remaining question is the following:

**Question 8.** Is it consistent that every Boolean $C(K)$ which has the Grothendieck property has the Nikodým property?

In Talagrand’s construction the algebra $A$ of clopen subsets of $K$ has the following property (*): there is a countable subalgebra $A_0 \subseteq A$ such that for any $a \in A$ there are $a_n \in A_0$ such that $\lambda(a_n \triangle a) \to 0$ (i.e., $\lambda$ has countable Maharam type) and the measures witnessing the failure of the Nikodým property are absolutely continuous with respect to $\lambda$. We do not know if this must be true for all algebras like in [51]. Thus an easier version of Question 7 would be:

**Question 9.** Is it consistent that every $C(K)$ for $K$ totally disconnected with property (*) does not have the Grothendieck property?

Requiring that a totally disconnected $K$ has both the Nikodým and Grothendieck properties is thus even a stronger way of imposing that $K$ has no convergent sequences. It turns out that it is equivalent to the known Vitali–Hahn–Saks property. For a Boolean algebraic and measure-theoretic treatment of these properties see [48].

One can multiply questions about $C(K)$s with the Grothendieck, Nikodým, or Vitali–Hahn–Saks property like questions on convergent sequences of compact spaces. For example, what about the weights of compact spaces without convergent sequences? We have many interesting topological results on it (see [13]) and just one $C(K)$ result of Brech ([7]) showing that it is consistent that there are $C(K)$s with the Grothendieck property of density less than the continuum. One could define cardinal invariants $\mathfrak{gr}$, $\mathfrak{ni}$, $\mathfrak{vhs}$ as minimal infinite densities of $C(K)$s ($K$ totally disconnected in the second and third case) which have Grothendieck, Nikodým, Vitali–Hahn–Saks properties respectively and $\mathfrak{ncs}$ could stand for the minimal infinite weight of a compact space without a converging sequence. We have $\mathfrak{p} \leq \mathfrak{ncs} \leq \mathfrak{gr}$, $\mathfrak{ni} \leq \mathfrak{vhs} \leq 2^\omega$ in ZFC and the result of Brech shows that $\mathfrak{gr} < 2^\omega$ is consistent.

**Question 10.** (a) What is the value of $\mathfrak{gr}$, $\mathfrak{ni}$, $\mathfrak{vhs}$ among known cardinal invariants of the continuum. In particular do we have in ZFC (b) $\mathfrak{ncs} = \mathfrak{gr}$?, (c) $\mathfrak{gr} = \mathfrak{p}$?

Another cardinal invariant of Boolean algebras which enters the game is the cofinality of a Boolean algebra. In [48, 4.6] it is shown that $\text{cf}(A) = \omega$ implies that $C(K)$ does not have the Nikodým nor the Grothendieck property where $K$ is the Stone space of $A$. 

Quotients
Not all quotients of $C(K)$ spaces are $C(L)$ for some $L$. For example, a nondispersed $K$ always has $\ell_2$ as a quotient ([34]). In some cases, like for Grothendieck $C(K)$ spaces, the only separable quotients are reflexive, thus the result saying that if $K$ has no convergent sequences then it has no infinite metric subspaces has its corresponding version that Grothendieck $C(K)$ spaces do not have infinite-dimensional separable quotients of the form $C(K)$.

The still-open Efimov problem for compact spaces is whether any infinite compact space contains a copy of a convergent sequence or a copy of $\beta\mathbb{N}$. It has negative answer (first obtained by Fedorchuk [17]) only under a special set-theoretic assumptions, but we do not know if a counterexample can be obtained in ZFC. However a $C(K)$ version of the problem is independent and actually may be suggesting the way of solving the topological version: Talagrand shows in [50] that under CH there is a Grothendieck $C(K)$ which doesn’t have $\ell_\infty$ as a quotient. On the other hand Haydon, Levy and Odell show in [21] that under $\mathfrak{p}=2^{\omega}\omega>2^\omega$ every $C(K)$ which is Grothendieck (i.e., doesn’t have $c_0$ as a quotient) has $\ell_\infty$ as a quotient.

One can even get in ZFC a Grothendieck $C(K)$ which has no subspace isomorphic to $\ell_\infty$ (see [20]). In this paper a lemma due to Argyros is proved which says that if a Boolean algebra has the subsequential completeness property (this implies the Grothendieck property and means that for any antichain $(a_n)$ in the algebra there is an infinite $M$ such that $\bigvee_{n\in M}a_n$ exists in the algebra), then it has an uncountable independent family.

**Question 11.** Is it consistent that there is a Boolean algebra with the subsequential completeness property without an independent family of cardinality $2^\omega$?

### Complemented subspaces

Combining quotients and subspaces we obtain complemented subspaces. A subspace $Y$ of a Banach space $X$ is said to be complemented in $X$ if and only if there is a bounded operator $P: X \to Y$, called a projection, onto $Y$ such that $P \restriction Y = \text{Id}_Y$. It is equivalent to the fact that $P^2 = P$ and to the existence of a decomposition of $X$ as $Y \oplus Z$. If $P$ is a projection, $Z = \text{Ker}(P)$ ([49, 12]). A topological operation corresponding to a complemented subspace is a retraction. If $F: K \to L$ is a retraction onto $L \subseteq K$ (i.e., $F \restriction L = \text{Id}_L$), then $P: C(K) \to C(K)$ is a norm-one projection where $P(f) = f \circ F$. The image of this projection is isometric to the $C(L)$, namely the restriction to $L$ is the isometry from $P[C(K)]$ onto $C(L)$. Projections play a more important role in Banach space theory than retractions in topology because they define decompositions of Banach spaces.

There are other canonical topological ways of defining complemented subspaces of $C(K)$ through the theory of averaging and extension operators whose classical period is depicted in Pełczyński’s monograph [41]. The work of Ditor, Haydon, Koppelberg, Ščepin and others contributed to resolving the main questions left after [41]. See the introductions of [5, 19, 29] for references and glimpses of this story.
We saw in the previous sections that trivially not all quotients nor subspaces of \( C(K) \) spaces are again of this form. The analogous fact about complemented subspaces of the \( C(K) \)'s is unknown (cf. \([47]\)).

**Question 12.** Is every complemented subspace of any \( C(K) \) of the form \( C(L) \)?

It is even not known in the case of \( C(2^\omega) \) or \( C([0, \alpha]) \) for \( \alpha \geq \omega^\omega \). For partial results in this metric case of \( K \) see \([47]\). The deepest result describing the complemented subspaces of the \( C(K) \)'s in general remains Pelczynski's theorem saying that such subspaces have isomorphic copies of \( c_0 \) (\([39]\, \text{Cor. 2}\)).

Now let us talk about the structure of complemented subspaces of a \( C(K) \).

A surprising result of \([30]\) is that there are \( C(K) \)'s which are indecomposable, i.e., whose only complemented subspaces are finite-dimensional or co-finite-dimensional (such subspaces are always complemented in any Banach space). This result is obtained by constructing \( K \) which admits few functions like \( T^* \), i.e., we are able to control all operators on the \( C(K) \). Such \( T^* \) are of the form \( gI + S \) where \( gI \) is a multiplication of measures by a Borel function \( g \) and \( S \) is a weakly compact operator. Assuming \( \text{CH} \) we can get \( K \) such that every operator \( T \) on \( C(K) \) is of the form \( gI + S \) where \( g \in C(K) \) (i.e., \( gI \) is the multiplication of functions by a continuous function \( g \)) and \( S \) is a weakly compact operator. In \([43]\), Plebanek removed the need of \( \text{CH} \) from the last statement at the price of losing the separability of \( K \) and his \( C(K) \) is not a subspace of \( \ell_\infty \).

**Question 13.** Is it true in ZFC that there is a \( C(K) \) which is a subspace of \( \ell_\infty \) where every operator is of the form \( gI + S \) where \( S \) is weakly compact and \( g \in C(K) \) ?

The results of \([30]\) show that Banach spaces of the type obtained by Gowers and Maurey may be of the form \( C(K) \). One construction however, the Schröder–Bernstein problem, is left:

**Question 14.** Are there compact \( K \) and \( L \) such that \( C(K) \) and \( C(L) \) are nonisomorphic but each is isomorphic to a complemented subspace of the other?

As the reader must have noted, in this section we entered the realm of operators on Banach spaces since being complemented is equivalent to the existence of some operator. Weakly compact operators which are not of finite-dimensional range are Banach spaces theory strangers in the land of \( K \)'s. However in any \( C(K) \) there is room for perturbing operators by a weakly compact operator, i.e., an operator which sends bounded sets to relatively weakly compact sets. As by Gantmacher’s theorem \( T \) is weakly compact if an only if \( T^* \) is, so the measures \( \mu_n = T^*(\delta_{x_n}) \) for such \( T \) must form a weakly compact set in \( M(K) \) for any sequence of points \( x_n \in K \). By the Dieudonné–Grothendieck characterization of such sets in \( M(K) \) the sup \( \mu_n(U_k) \) always goes to 0 when \( k \to \infty \) for a pairwise disjoint sequence \((U_n)\) of open subsets of \( K \). So, indeed if \( T \) is weakly compact, then one cannot recover even from \( T^* \) any part reasonable in terms of mappings of \( K \). A researcher of topological origin should feel more secure facing the weakly
compact perturbations in $C(K)$s after seeing the list in [10] of properties of operators in $C(K)$ equivalent to weak compactness from which we mention just one: $T$ on $C(K)$ is weakly compact if and only if it is not an isomorphism while restricted to any infinite-dimensional subspace, i.e., $T$ is strictly singular ([40]).

In [32] we pushed the construction of a Boolean $C(K)$ with few operators to densities above $2^{\omega}$. This strongly answered in the negative the question whether any Banach space can have complemented subspaces of density $\leq 2^{\omega}$ above any separable subspace. The issues of the densities of complemented subspaces are excellently surveyed in [45] where several open problems are stated. However our $K$ of [32] exists only consistently. Immediate questions which appear are:

**Question 15.** Is it consistent that any Banach space has a complemented subspace of density $\leq 2^\omega$?

S. Argyros suggested the following:

**Question 16.** Does there exist any bound for the density of indecomposable Banach spaces?

This is quite natural if one remembers that hereditarily indecomposable Banach spaces may have density at most $2^\omega$ ([45]). Related topological questions also seem unanswered:

**Question 17.** Is there in ZFC a compact space without infinite retracts of weights $\leq 2^\omega$?

The methods of [32] suggest that if its results are not true in ZFC, then large cardinals (cf. [24]) may provide tools necessary for obtaining the other consistency. As far as now we only know how to get decomposable subspaces of large Banach spaces ([27]), but no methods on decomposing entire large spaces seem available.

In [31] assuming CH, we proved that there is a scattered space $K$ with a minimal space of operators, i.e., where every operator is of the form $cI + S$ where $c$ is a real and $S$ has its range included in a copy of $c_0$. This has deep implications with respect to the complemented subspaces.

**Question 18.** Is there in ZFC a compact nonmetrizable scattered $K$ such that all operators on $C(K)$ are of the form $cI + S$ where $c$ is a real and $S$ has its range included in a copy of $c_0$?

Argyros ([3]) constructed a separable nonisomorphic to $c_0$ (thus not a $C(K)$) Banach space $X$ whose only decompositions are $c_0 \oplus X$.

Besides asking about complemented subspaces we may inquire about being complemented in superspaces. A Banach space is said to be injective if it is complemented in any superspace. All finite-dimensional spaces are injective by the Hahn–Banach theorem. Similarly $\ell_\infty \equiv C(\beta\mathbb{N})$ is injective; just extend the coordinate functionals by the Hahn–Banach theorem. For equivalent definitions, see [37]. One of them is that $X$ is injective if and only if whenever $Z \supseteq Y$ are Banach spaces, $T: Y \to X$ is a bounded operator, then there is an extension $T': Z \to X$ of $T$. 
Another way to prove (see [37]) that a \( C(K) \) is injective is to construct a projection from \( \ell_\infty(K) \) (the space of all bounded not necessarily continuous functions on \( K \)) onto \( C(K) \). If \( K \) is extremally disconnected (i.e., the Stone space of a complete Boolean algebra), one constructs such a projection by the Boolean algebraic Sikorski extension criterion ([28]). The stakes are high in the following question, namely knowing the injective objects in the category of Banach spaces with their isomorphisms:

**Question 19.** If a Banach space \( C(K) \) is injective is it isomorphic to a \( C(L) \) where \( L \) is extremally disconnected?

Much effort was done to settle this question in the 1970s but the results are very partial. Grothendieck ([18]) proved that injective \( C(K) \)s are Grothendieck; in particular, \( K \)s have no convergent sequences, Amir ([1]) proved that such a \( K \) contains a dense open extremally disconnected subset. Rosenthal’s results together with Pelczyński’s decomposition method imply that a ccc \( K \) such that \( C(K) \) is injective and not isomorphic to \( \ell_\infty \) cannot be separable and Wolfe ([54, 55]) proved that such \( K \)s must be totally disconnected and a union of finitely many extremely disconnected (not necessarily compact) spaces.

One of the problems is how to prove that a \( C(K) \) is not isomorphic to any \( C(L) \) for \( L \) extremally disconnected. We do not have strong isomorphic properties of such \( C(L) \)s other than the Grothendieck property which can be shared by very different spaces ([7, 20, 50]). One shouldn’t be also discouraged to try a positive result. If a Banach space is 1-complemented in any Banach space (i.e., the projection is of norm one), then Goodner, Kelley and Nachbin managed to prove that it is isomorphic to a \( C(K) \) for \( K \) extremally disconnected (see [37]).

We have to mention at the end the following:

**Question 20.** Is it consistent that \( \ell_\infty/c_0 \equiv C(\omega^*) \) is isomorphic to \( A \oplus B \) and none of the spaces \( A \) nor \( B \) is isomorphic to \( \ell_\infty/c_0 \)?

This is related to a result of Drewnowski and Roberts [14] which says that it is impossible under \( CH \). Again, an information on complemented subspaces of \( C(\omega^*) \) is obtained by conquering some partial knowledge about all operators on \( C(\omega^*) \). S. Todorčević suggested that one could develop a theory of operators on \( C(\omega^*) \) corresponding to the theory of autohomeomorphisms of \( \omega^* \).

If the answer to the above question were positive, it would mean that \( \ell_\infty/c_0 \) may fail to have the Schroder–Bernstein property as suggested in [8], as \( \ell_\infty/c_0 \) must be complemented in one of the spaces \( A \) or \( B \) by the results of [14].

For \( p \in K \) define \( C_0(K,p) \) to be the set of all functions in \( C(K) \) which are zero in \( p \). A natural decomposition for answering Question 20 would be of the form \( (R \oplus C_0(X,p)) \oplus C_0(Y,p) \) where \( p \in \omega^* \) and \( X, Y \) are open subsets of \( \omega^* \) that would satisfy the conditions of the following:

**Question 21.** Is it consistent that there are \( p \in \omega^* \) and open \( X, Y \subseteq \omega^* \) such that \( X \cap Y = \emptyset \), \( \{p\} = \overline{X} \cap \overline{Y} \) and none of the \( X \) nor \( Y \) is homeomorphic to \( \omega^* \)?
To answer Question 20 it would be sufficient to prove that $C(\overline{X})$ and $C(\overline{Y})$ are not isomorphic to $l_\infty/c_0$ instead of $\overline{X}$ and $\overline{Y}$ not being homeomorphic to $\omega^*$. This follows from the fact that

$$C(Z) \sim R \oplus C_0(Z, p) \sim C_0(Z, p)$$

for any open $Z \subseteq \omega^*$, since $Z$ must contain a copy of $\beta N$, $C(\beta N) \sim l_\infty \sim l_\infty \oplus R$ and $l_\infty$ is complemented in any superspace as an injective Banach space.

Most of the issues, even these set-theoretic topological, in the isomorphic theory of the $C(K)$ were not mentioned in this article. For example the questions of unconditional, transfinite and Markushevich’s bases, biorthogonal and semibiorthogonal sequences, irredundant sets in Boolean algebras, or the topology of the dual ball involving such questions as countable tightness or hereditary separability or hereditary Lindelöf degree. Analogously the weak topology of the $C(K)$, its relation to the pointwise convergence topology with its vast literature and open problems has been untouched.

Acknowledgement. Many thanks to A. Aviles, O. Kalenda, W. Marciszewski, G. Plebanek for trying to explain to the author some facts on $C(K)$'s related to this article.

References


§52. Koszmider, *Compact spaces and their Banach spaces of continuous functions*


Tightness and $t$-equivalence

Oleg Okunev

All spaces below are assumed to be Tychonoff (that is, completely regular Hausdorff). We use terminology and notation as in [5], with the exception that the tightness of a space $X$ is denoted as $t(X)$.

Two spaces $X$ and $Y$ are called $M$-equivalent if their free topological groups $F(X)$ and $F(Y)$ in the sense of Markov [7] are topologically isomorphic. The spaces $X$ and $Y$ are $l$-equivalent if the spaces $C_p(X)$ and $C_p(Y)$ of real-valued continuous functions equipped with the topology of pointwise convergence are linearly homeomorphic, and $t$-equivalent if $C_p(X)$ and $C_p(Y)$ are homeomorphic (see [3]); Arhangel’skiǐ showed in [2] that $M$-equivalence of two spaces implies their $l$-equivalence; clearly, $l$-equivalent spaces are $t$-equivalent. We say that a topological property is preserved by an equivalence relation if whenever two spaces are in the relation, one of them has the property if and only if the other one does. Similarly, we say that a cardinal invariant is preserved by a relation if its values on two spaces are the same whenever the spaces are in the relation.

The article [9] contains an example that shows that the sequentiality and tightness are not preserved by the relation of $M$-equivalence. Tkachuk proved in [15] that the tightness is preserved by $l$-equivalence in the class of compact spaces, that is, if $X$ and $Y$ are $l$-equivalent compact spaces; this was later extended in [12] by showing that the tightness is preserved by $t$-equivalence in the class of compact spaces, and the same holds for sequentiality if $2^{t} > \mathfrak{c}$ (in fact it easily follows from the main theorem in [12] that if $X$ and $Y$ are $t$-equivalent spaces and $X$ is a countable union of its compact sequential subspaces, then so is $Y$).

As for the Fréchet property, it is not preserved by the relation of $M$-equivalence even in the class of compact spaces [13].

The example in [9] that shows the non-preservation of the tightness by $M$-equivalence depends heavily on the fact that one of the two spaces is not normal. Indeed, the construction of the example uses the fact that if $K$ is a retract of a space $X$, then the spaces $X^+$ obtained by adding an isolated point to $X$ and the direct sum of the spaces $K$ and $X/K$ are $M$-equivalent [9]; here $X/K$ is the partition of $X$ whose elements are $K$ and singletons equipped with the R-quotient topology (that is, the strongest completely regular topology that makes the natural mapping $p: X \to X/K$ continuous). It is easy to see that if $X$ is normal, then the natural mapping $p: X \to X/K$ is in fact quotient, and therefore, closed; by Theorem 4.5 in [1], in this case the tightness of the space $X$ is equal to the supremum of the tightnesses of the image space $X/K$ and of the fibers of $p$; the only nontrivial fiber of $p$ is $K$, so $t(X^+) = t(X) = t(X/K \oplus K)$.

Hence, the following question:

**Problem 1.** Let $X$ and $Y$ be normal $M$-equivalent spaces. Is it true that $t(X) = 1197? t(Y)$?
Since $t$-equivalent compact spaces have the same tightness, it is natural to ask whether this fact may be generalized to a wider class of spaces.

**Problem 2.** Let $X$ and $Y$ be $M$-equivalent $\sigma$-compact spaces. Is it true that $t(X) = t(Y)$?

Similar questions remain open for the relations of $l$-equivalence and $t$-equivalence.

It is known that compactness is preserved by $l$-equivalence (and hence by $M$-equivalence) [17], but not by $t$-equivalence [6]. Thus, the following question is specific for the relation of $t$-equivalence:

**Problem 3.** Let $X$ and $Y$ be $t$-equivalent spaces such that $X$ is compact. Is it true that $t(X) = t(Y)$?

Note that every space $t$-equivalent to a compact space is $\sigma$-compact [8]. It can easily be deduced from the main theorem in [12] that if $X$ is compact (in fact, the Lindelöf property is sufficient) and $Y$ is $t$-equivalent to $X$, then every free sequence in $Y$ is of length at most $t(X)$; unfortunately, for a $\sigma$-compact space $Y$ this is not sufficient to conclude that the tightness of $Y$ is at most $t(X)$ [14].

Of course, the following versions of Problems 1 and 2 quite naturally arise:

**Problem 4.** Let $X$ and $Y$ be $M$-equivalent Lindelöf spaces. Is it true that $t(X) = t(Y)$?

**Problem 5.** Let $X$ and $Y$ be $M$-equivalent paracompact spaces. Is it true that $t(X) = t(Y)$?

as well as their versions for the relations of $l$-equivalence and of $t$-equivalence.

Unlike the compact case, the tightness of $\sigma$-compact spaces is not productive (see, e.g., [11, 16]). There may be more hope for positive answers (or more challenge for finding examples) for the versions of the above problems where the equality $t(X) = t(Y)$ is replaced by $t(Y) \leq t^*(X) = \sup \{t(X^n) : n \in \omega\}$. The example in Section 2 in [10] shows that there are $M$-equivalent spaces $X$ and $Y$ where $X$ is metrizable and $Y$ has one nonisolated point such that the tightness of $Y^2$ is uncountable, so we cannot expect $t^*(X) = t^*(Y)$ for paracompact spaces.

While [12] contains the proof that there is a topological property of $C_p(X)$ that, assuming that $X$ is compact, is equivalent to the countability of the tightness of $X$, there is no internal description of this property. Hence, the following (somewhat fuzzy) request:

**Problem 6.** Find an internally defined cardinal function $\phi$ such that whenever $X$ is a compact space, $t(X) = \phi(C_p(X))$.

The inequality $t(Y) \leq t(X)$ for compact $X$ and $Y$ is proved in [12] under the assumption that $C_p(Y)$ is an image under an open mapping of a subspace of $C_p(X)$, so we might expect that the function $\phi$ in Problem 6 should be hereditary and not raise in continuous open images. The following modification of Problem 6 for the relation of $l$-equivalence also appears interesting:
Problem 7. Find a cardinal function $\phi$ internally defined for locally convex linear topological spaces such that whenever $X$ is a compact space, $t(X) = \phi(C_p(X))$.

A similar problem arises for the sequentiality:

Problem 8. Find an internally defined topological property $P$ such that whenever $X$ is a compact space, $X$ is a countable union of closed sequential subspaces iff $C_p(X)$ has $P$.

Tkachuk essentially proved in [15] that a compact space $X$ is a countable union of closed sequential subspaces iff so is $L_p(X)$, and a compact $X$ has countable tightness iff $L_p(X)$ is a countable union of its subspaces of countable tightness; here $L_p(X)$ is the weak dual space of $C_p(X)$.

An interesting hypothesis related to Problem 6 was communicated by E. Reznichenko:

Problem 9. Let $X$ be a compact space of countable tightness. Is it true that every compact subspace of $C_p(C_p(X))$ has countable tightness?

The negative answer to the next question would give a consistently positive answer to Problem 9 (for example, in the model of ZFC described in [4]).

Problem 10. Is there a compact space of countable tightness $X$ such that $\omega_1 + 1$ (with the order topology) is homeomorphic to a subspace of $C_p(C_p(X))$?

References


Topological problems in nonlinear and functional analysis

Biagio Ricceri

In this note, I intend to collect some problems, conjectures and perspectives, of topological nature, arising from the research work I made in the last years.

I start recalling the following definition.

Let \((E, \| \cdot \|)\) be a real normed space. A nonempty set \(A \subset E\) is said to be antiproximinal with respect to \(\| \cdot \|\) if, for every \(x \in E \setminus A\) and every \(y \in A\), one has \(\|x - y\| > \inf_{z \in A} \|x - z\|\).

I then propose the following

Conjecture 1. There exists a noncomplete real normed space \(E\) with the following property: for every nonempty convex set \(A \subset E\) which is antiproximinal with respect to each norm on \(E\), the interior of the closure of \(A\) is nonempty.

The main reason for the study of Conjecture 1 is to give a contribution to open mapping theory in the setting of noncomplete normed spaces. Recall that in any vector space there exists the strongest vector topology of the space [30, p. 42]. Actually, making use of Theorem 4 of [41], one can prove the following result.

Theorem 1. Let \(X, E\) be two real vector spaces, \(C\) a nonempty convex subset of \(X\), \(F\) a multifunction from \(C\) onto \(E\), with nonempty values and convex graph. Then, for every nonempty convex set \(A \subseteq C\) which is open with respect to the relativization to \(C\) of the strongest vector topology on \(X\), the set \(F(A)\) is antiproximinal with respect to each norm on \(E\).

Now, I am going to present a problem about an unusual way of finding global minima of functionals in Banach spaces. A closed hyperplane in a real normed space \(X\) is any set of the type \(T^{-1}(r)\), where \(T\) is a nonzero continuous linear functional on \(X\) and \(r \in \mathbb{R}\). First, I recall the following result from [48] (see also [43, 46, 49, 50]):

Theorem 2 ([48, Theorem 2.1]). Let \((T, \mathcal{F}, \mu)\) be nonatomic measure space, with \(\mu(T) < +\infty\), \(E\) a real Banach space, and \(f: E \to \mathbb{R}\) a bounded below Borel functional such that, for some \(\gamma \in ]0, 1[\),

\[
\sup_{x \in E} \frac{f(x)}{\|x\|^\gamma + 1} < +\infty.
\]

Then, for every \(p \geq 1\) and every closed hyperplane \(V\) of \(L^p(T, E)\), one has

\[
\inf_{u \in V} \int_T f(u(t))d\mu = \inf_{u \in L^p(T, E)} \int_T f(u(t))d\mu.
\]

I then propose the following
Problem 1. Let $X$ be an infinite-dimensional real Banach space, and let $J: X \to \mathbb{R}$ be a bounded below functional satisfying, for some $\gamma \in ]0, 1[$,
\[
\sup_{u \in X} \frac{J(u)}{\|u\|^\gamma + 1} < +\infty.
\]
Find conditions under which there exists a closed hyperplane $V$ of $X$ such that the restriction of $J$ to $V$ has a local minimum.

The motivation for the study of Problem 1 is as follows. Assume that we wish to minimize a bounded below Borel functional $f$ on a real Banach space $E$ satisfying the condition, for some $\gamma \in ]0, 1[$,
\[
\sup_{x \in E} \frac{f(x)}{\|x\|^\gamma + 1} < +\infty.
\]
For each $u \in L^1([0, 1], E)$, set
\[
J(u) = \int_0^1 f(u(t))dt.
\]
So, $J$ is bounded below and satisfies $(\ast)$ with $X = L^1([0, 1], E)$. Assume that there is some closed hyperplane $V$ of $L^1([0, 1], E)$ such that the restriction of $J$ to $V$ has a local minimum, say $u_0$. By a result of Giner ([28]), $u_0$ is actually a global minimum of the restriction of $J$ to $V$. On the other hand, by Theorem 2, we have
\[
\inf_{u \in V} J(u) = \inf_{u \in L^1([0, 1], E)} J(u)
\]
and so $u_0$ is a global minimum of $J$ in $L^1([0, 1], E)$. This easily implies that $f$ has a global minimum in $E$.

Now, I list a series of possible new proofs of the Brouwer fixed point theorem recalling the results from which they originate. $\langle \cdot, \cdot \rangle$ will denote the usual inner product in $\mathbb{R}^n$.

Problem 2. Let $X \subset \mathbb{R}^n$ ($n \geq 2$) be a compact convex set and $f: X \to X$ a continuous function. Without using any result based on the Brouwer fixed point theorem, is it possible to find a continuous function $\alpha: X \to \mathbb{R}$ in such a way that the set $\{(x, y) \in X \times \mathbb{R}^n : \langle f(x) - x, y \rangle = \alpha(x)\}$ is disconnected?

A positive answer to Problem 2 would provide a new proof of the Brouwer theorem via the following results ([45]; see also [37]):

**Theorem 3** ([45, Theorem 2]). Let $X$ be a topological space, let $E$ be a real topological vector space (with topological dual $E^\ast$), and let $A: X \to E^\ast$ be such that the set $\{y \in E : x \mapsto \langle A(x), y \rangle \text{ is continuous}\}$ is dense in $E$. Then, the following assertions are equivalent: (i) The set $\{(x, y) \in X \times E : A(x)(y) = 1\}$ is disconnected. (ii) The set $X \setminus A^{-1}(0)$ is disconnected.

**Proposition 1** ([45, Proposition 1]). Let $X$ be a topological space, let $E$ be a real topological vector space (with algebraic dual $E'$) and let $A: X \to E'$. Assume that, for some continuous function $\alpha: X \to \mathbb{R}$, the set $\{(x, y) \in X \times E : A(x)(y) = \alpha(x)\}$
is disconnected. Then, either $A^{-1}(0) \neq \emptyset$ or the set \{(x, y) \in X \times E : A(x)(y) = 1\} is disconnected.

Assume that Problem 2 admits a positive answer. Apply Proposition 1 taking $E = \mathbb{R}^n$ and $A(x) = f(x) - x$ for all $x \in X$ (of course, $E'$ is identified with $\mathbb{R}^n$). Since $X$ is connected, Proposition 1 (on the basis of Theorem 3) ensures that $A$ has a zero, that is $f$ has a fixed point.

A possible positive answer to Problem 2 could be rather difficult due to the fact that the function $\alpha$ does not depend on $y$. I then propose a variant of Problem 2 without such a restriction.

**Problem 3.** Let $X \subset \mathbb{R}^n$ ($n \geq 2$) be a compact convex set and $f : X \to X$ a continuous function. Without using any result based on the Brouwer fixed point theorem, is it possible, for each $\epsilon > 0$, to find a continuous function $\alpha_\epsilon : X \times \mathbb{R}^n \to \mathbb{R}$, with $\alpha_\epsilon(x, \cdot)$ Lipschitzian in $\mathbb{R}^n$ with Lipschitz constant less than or equal to $\epsilon$, in such a way that the set \{(x, y) \in X \times \mathbb{R}^n : (f(x) - x, y) = \alpha_\epsilon(x, y)\} is disconnected?

Problem 3 originates from the following

**Theorem 4 ([47, Theorem 19]).** Let $X$ be a connected topological space, $E$ a real Banach space, $A$ an operator from $X$ into $E^*$, $\alpha$ a real function on $X \times E$ such that, for each $x \in X$, $\alpha(x, \cdot)$ is Lipschitzian in $E$, with Lipschitz constant $L(x) \geq 0$. Further, assume that the set $\{y \in E : A(\cdot)(y) - \alpha(\cdot, y) \text{ is continuous} \}$ is dense in $E$ and that the set \{(x, y) \in X \times E : A(x)(y) = \alpha(x, y)\} is disconnected. Then, there exists some $x_0 \in X$ such that $\|A(x_0)\|_{E^*} \leq L(x_0)$.

Arguing as before, a positive answer to Problem 3 would produce a new proof of the Brouwer theorem via Theorem 4 and an approximation argument.

I also wish to propose the following

**Conjecture 2.** Let $X \subset \mathbb{R}^n$ ($n \geq 2$) be a compact convex set and $f : X \to X$ a continuous function. Let $\epsilon > 0$ be small enough. Denote by $\Lambda_\epsilon$ the set of all continuous functions $\alpha : X \times \mathbb{R}^n \to \mathbb{R}$ such that, for each $x \in X$, $\alpha(x, \cdot)$ is Lipschitzian in $\mathbb{R}^n$, with Lipschitz constant less than or equal to $\epsilon$. Consider $\Lambda_\epsilon$ equipped with the relativization of the strongest vector topology on the space $\mathbb{R}^{X \times \mathbb{R}^n}$. Then, the set $\{(\varphi, x, y) \in \Lambda_\epsilon \times X \times \mathbb{R}^n : (f(x) - x, y) = \alpha(x, y)\}$ is disconnected.

On the basis of Theorem 5 below, it would be of interest to prove Conjecture 2 without using the Brouwer theorem.

**Theorem 5 ([47, Theorem 21]).** Let $X$ be a connected and locally connected topological space, $E$ a real Banach space, $A : X \to E^*$ a continuous operator with closed range. For each $\epsilon > 0$, denote by $\Lambda_\epsilon$ the set of all continuous functions $\alpha : X \times E \to \mathbb{R}$ such that, for each $x \in X$, $\alpha(x, \cdot)$ is Lipschitzian in $E$, with Lipschitz constant less than or equal to $\epsilon$. Consider $\Lambda_\epsilon$ equipped with the relativization of the strongest vector topology on the space $\mathbb{R}^{X \times E}$, and assume that the set $\{(\alpha, x, y) \in \Lambda_\epsilon \times X \times E : A(x)(y) = \alpha(x, y)\}$ is disconnected. Then, $A^{-1}(0) \neq \emptyset$. 

To introduce the last problem related to the Brouwer theorem, let me recall a further result. The spaces $C^0(X, E)$ and $C^0(X)$ that will appear are considered with the sup-norm. Recall that a subset $D$ of a topological space $S$ is a retract of $S$ if there exists a continuous function $h: S \to D$ such that $h(s) = s$ for all $s \in D$.

**Theorem 6 ([55, Theorem 6]).** Let $X$ be a compact Hausdorff topological space, $E$ a real Banach space, with $\dim(E) \geq 2$, and $A: X \to E^*$ a continuous operator. Then, at least one of the following assertions holds: (a) $A^{-1}(0) \neq \emptyset$. (b) There exists $\epsilon > 0$ such that, for every Lipschitzian operator $J : C^0(X, E) \to C^0(X)$, with Lipschitz constant less than $\epsilon$, the set \{$\psi \in C^0(X, E) : A(x)(\psi(x)) = J(\psi)(x)$ for all $x \in X$\} is an unbounded retract of $C^0(X, E)$.

Theorem 6 gives the motivation for the following

**Problem 4.** Let $X \subset \mathbb{R}^n$ ($n \geq 2$) be a compact convex set and $f : X \to X$ a continuous function. Without using any result based on the Brouwer fixed point theorem, is it possible to prove that, for each $\epsilon > 0$, there exists a Lipschitzian operator $J : C^0(X, \mathbb{R}^n) \to C^0(X)$, with Lipschitz constant less than $\epsilon$, in such a way that the set \{$\psi \in C^0(X, \mathbb{R}^n) : \langle f(x) - x, \psi(x) \rangle = J(\psi)(x)$ for all $x \in X$\} is either bounded or disconnected?

Before formulating the next conjecture, I recall two more results:

**Theorem 7 ([42, Theorem 2.2]).** Let $X$ be a topological space and let $S \subseteq X \times [0, 1]$ be a connected set whose projection on $[0, 1]$ is the whole of $[0, 1]$. Then, $S$ intersects the graph of any continuous function from $X$ into $[0, 1]$.

**Theorem 8 ([50, Proposition 2.1]).** Let $X$ be a normed space, let $T \in X^*$ and let $J : X \to \mathbb{R}$ be a Lipschitzian functional with Lipschitz constant $L < \|T\|_{X^*}$. Then, the functional $T + J$ is onto $\mathbb{R}$.

I now state

**Conjecture 3.** Let $(X, \langle \cdot, \cdot \rangle)$ be an infinite-dimensional real Hilbert space and let $A : [0, 1] \to X$ be a continuous function such that, for some $\lambda \in [0, 1]$, one has

$$\sup_{x \in X} \inf_{t \in [0, 1]} (\langle A(t), x \rangle - \lambda \|A(t)\| \|x\|) < +\infty.$$  

Then, there exist $\mu \in [0, 1]$ and a continuous function $g : X \to [0, 1]$ such that

$$\sup_{x \in X} (\langle A(g(x)), x \rangle - \mu \|A(g(x))\| \|x\|) < +\infty.$$  

The motivation for the study of Conjecture 3 is as follows. Assume that it is true. I then claim that $A^{-1}(0) \neq \emptyset$. Indeed, put

$$M := \sup_{x \in X} (\langle A(g(x)), x \rangle - \mu \|A(g(x))\| \|x\|).$$

Consider the set

$$S := \{(t, x) \in [0, 1] \times X : \langle A(t), x \rangle = \mu \|A(t)\| \|x\| + M + 1\}.$$
If \( t \in [0, 1] \) does not belong to the projection of \( S \) on \([0, 1] \), then, in view of Theorem 8, we clearly have \( A(t) = 0 \), and we are done. Therefore, assume that such a projection is the whole of \([0, 1] \). In this case, we observe that \( S \) does not meet the graph of the continuous function \( g \), and so, by Theorem 7, \( S \) must be disconnected. At this point, we can apply Theorem 4 which ensures the existence of \( t_0 \in [0, 1] \) such that \( \|A(t_0)\| \leq \mu \|A(t_0)\| \). Since \( \mu < 1 \), one then has \( A(t_0) = 0 \), and the claim is proved.

Let me now recall the notion of Gâteaux differentiability. Let \( X \) be a real normed space. A functional \( J: X \to \mathbb{R} \) is said to be Gâteaux differentiable at a point \( x \) if there is \( T \in X^* \) such that
\[
\lim_{\lambda \to 0^+} \frac{J(x + \lambda y) - J(x)}{\lambda} = T(y)
\]
for all \( y \in X \). The functional \( T \) is the Gâteaux derivative of \( J \) at \( x \) and is denoted by \( J'(x) \). The functional \( J \) is said to be of class \( C^1 \) if it is Gâteaux differentiable at any point of \( X \) and the operator \( J': X \to X^* \) is continuous. The critical points of \( J \) are the zeros of \( J' \). Of course, if, for some vector topology on \( X \), the point \( x \) is a local minimum of \( J \) and \( J \) is Gâteaux differentiable at \( x \), then \( J'(x) = 0 \).

The following problem seems to be fascinating.

**Problem 5.** Let \( X \) be a real Banach space and let \( J: X \to \mathbb{R} \) be a functional of class \( C^1 \). Is there a topology \( \tau \) on \( X \) such that the critical points of \( J \) are exactly the \( \tau \)-local minima of \( J \)?

Clearly, a possible positive answer to Problem 5 would be of great theoretical interest. At this point, I would like to point that in [51–53, 56, 58] one can find various results on local minima that have been widely applied to nonlinear differential equations (see, for instance, [1–23, 25–27, 29, 31–36, 38, 39, 54, 59, 60]).

In [57], I got the following general result (see also [24]):

**Theorem 9** ([57, Theorem 2]). Let \( X \) be a real Hilbert space and let \( J: X \to \mathbb{R} \) be a nonconstant functional of class \( C^1 \), with compact derivative, such that
\[
\limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \leq 0.
\]
Then, for each \( r \in [\inf_X J, \sup_X J] \) for which the set \( J^{-1}([r, +\infty[) \) is not convex and for each convex set \( S \subseteq X \) dense in \( X \), there exist \( x_0 \in S \cap J^{-1}([-\infty, r[) \) and \( \lambda > 0 \) such that the equation \( x = \lambda J'(x) + x_0 \) has at least three solutions.

On the basis of Theorem 9, I now propose

**Problem 6.** Let \( X, Y \) be two topological spaces and let \( f: X \to Y \) be a continuous function. Assume that there is an open cover \( \mathcal{F} \) of \( X \) such that \( \text{card}(f^{-1}(y) \cap A) \leq 2 \) for all \( y \in Y \), \( A \in \mathcal{F} \). Find sufficient conditions in order that \( \text{card}(f^{-1}(y)) \leq 2 \) for all \( y \in Y \).

Here is the meaning of Problem 6. Let \( (P) \) be such a sufficient condition (concerning \( f \)). Let \( J \) satisfy the assumptions of Theorem 9 and let \( J^{-1}([r, +\infty[) \) be nonconvex for some \( r \in [\inf_X J, \sup_X J] \). Moreover, assume that, for each
\( \lambda > 0 \), there is some open cover \( \mathcal{F} \) of \( X \) such that, for every \( y \in X \) and every \( A \in \mathcal{F} \), the equation \( x = \lambda J'(x) + y \) has at most two solutions in \( A \). Then, for some \( \lambda > 0 \), the operator \( x \mapsto x - \lambda J'(x) \) does not satisfy condition \((P)\). The conclusion, of course, is a direct consequence Theorem 9. Clearly, the interest of results of this kind fully depends on the quality of the answers given to Problem 6.

In the final part of this note, I wish to propose some specific topological problems on the energy functional associated to the Dirichlet problem \((P_f)\)

\[ -\Delta u = f(x, u) \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0. \]

So, let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 3 \)) be an open bounded set. Let \( X = W^{1,2}_0(\Omega) \), with the usual norm \( \|u\| = (\int_\Omega |\nabla u(x)|^2 dx)^{\frac{1}{2}} \). For \( q > 0 \), denote by \( A_q \) the class of all Carathéodory functions \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) such that

\[
\sup_{(x,\xi) \in \Omega \times \mathbb{R}} \frac{|f(x,\xi)|}{1 + |\xi|^q} < +\infty.
\]

For \( 0 < q \leq \frac{n+2}{n-2} \) and \( f \in A_q \), put

\[
J_f(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx - \int_\Omega \left( \int_0^{u(x)} f(x, \xi) d\xi \right) dx
\]

for all \( u \in X \).

So, the functional \( J_f \) is of Class \( C^1 \) on \( X \) and one has

\[
J'_f(u)(v) = \int_\Omega \nabla u(x) \nabla v(x) dx - \int_\Omega f(x, u(x)) v(x) dx
\]

for all \( u, v \in X \). Hence, the critical points of \( J_f \) in \( X \) are exactly the weak solutions of problem \((P_f)\).

To formulate the next problem, denote by \( \tau_s \) the topology on \( X \) whose members are the sequentially weakly open subsets of \( X \). That is, a set \( A \subseteq X \) belongs to \( \tau_s \) if and only if for each \( u \in A \) and each sequence \( \{u_n\} \) in \( X \) weakly convergent to \( u \), one has \( u_n \in A \) for all \( n \) large enough.

**Problem 7.** Is there some \( f \in A_q \), with \( q < \frac{n+2}{n-2} \), such that, for each \( \lambda > 0 \) and \( r \in \mathbb{R} \), the functional \( J_{\lambda f} \) is unbounded below and the set \( J_{\lambda f}^{-1}(r) \) has no isolated points with respect to the topology \( \tau_s \)?

The interest for the study of Problem 7 comes essentially from the following result:

**Theorem 10** (\([53, \text{Theorem 3}]\)). Let \( f \in A_q \) with \( q < \frac{n+2}{n-2} \). Then, there exists some \( \lambda^* > 0 \) such that the functional \( J_{\lambda^* f} \) has local minimum with respect to the topology \( \tau_s \).

In the light of Theorem 10, the relevance of Problem 7 is clear. Actually, if \( f \) was answering Problem 7 in the affirmative, then, by Theorem 10, for some \( \lambda^* > 0 \), the functional \( J_{\lambda^* f} \) would have infinitely many local minima in the topology \( \tau_s \). Consequently, problem \((P_{\lambda^* f})\) would have infinitely many weak solutions.
It is also worth noticing that if \( f \in \mathcal{A}_q \) with \( q < \frac{n+2}{n-2} \) and \( \lim_{\|u\| \to +\infty} J_f(u) = +\infty \), then the local minima of \( J_f \) in the strong and in the weak topology of \( X \) do coincide ([40, Theorem 1]). On the other hand, if \( f(x, \xi) = |\xi|^{n-1}\xi \) with \( 1 < q < \frac{n+2}{n-2} \), then, for some constant \( \lambda > 0 \), it turns out that 0 is a local minimum of \( J_{\lambda f} \) in the strong topology but not in the weak one ([40, Example 2]). However, I do not know any example of \( f \) for which \( J_f \) has a local minimum in the strong topology but not in \( \tau_s \).

I conclude presenting what I consider the most important of the problems of this note.

**Problem 8.** Denote by \( \tau \) the strongest vector topology of \( X \). Is there some \( f \in \mathcal{A}_{\frac{n+2}{n-2}} \) such that the set \( \{(u, v) \in X \times X : J'_f(u)(v) = 1\} \) is disconnected in \( (X, \tau) \times (X, \tau) \)?

Assume that \( f \in \mathcal{A}_{\frac{n+2}{n-2}} \) have the property required in Problem 8. Since \( J_f \) is of class \( C^1 \), clearly the operator \( J'_f \colon X \to X^* \) is \( \tau \)-weakly-star continuous. Hence, by Theorem 3, the set \( X \setminus (J'_f)^{-1}(0) \) is \( \tau \)-disconnected. Then, this implies, in particular, that the set \( (J'_f)^{-1}(0) \) is not \( \tau \)-relatively compact ([44, Proposition 3]), and hence is infinite. So, for such an \( f \), problem \((P_f)\) would have infinitely many weak solutions. Hence, a possible positive answer to Problem 8 would open a completely new chapter in the theory of the multiplicity of solutions for problem \((P_f)\).

Finally, in personal communication T.-C. Tang has remarked that if \( f \in \mathcal{A}_q \) for some \( q < \frac{n+2}{n-2} \), then \( f \) cannot satisfy the property required in Problem 8. In other words, Problem 8 concerns nonlinearities with critical growth.

**References**

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References


Twenty questions on metacompactness in function spaces

V.V. Tkachuk

1. Introduction

A space $Z$ is called *metacompact* (or *weakly paracompact*) if any open cover of $Z$ has a point-finite refinement. This notion is well-known and thoroughly studied. It is preserved by closed maps and closed subspaces and coincides with paracompactness in the class of collectionwise normal spaces; besides, any pseudocompact metacompact space is compact (see the survey of Burke [4] for proofs and detailed treatment). Unfortunately, the list of important results on metacompactness is too long to be presented here.

However, the importance of metacompactness in general topology is not reflected in $C_p$-theory at all; the results are scarce and almost nothing can be said even if we ask the most naïve questions about metacompactness in $C_p(X)$. The purpose of this paper is to draw attention to a significant amount of interesting open problems as well as to numerous possibilities of a breakthrough in this area.

The material of this paper is presented in Section 3 and Section 4 which cover the general case and the compact case respectively. In fact, every problem, formulated in Section 3, is open for compact spaces as well. However, its clone is formulated in Section 4 (this occurs four times) only if it is of special importance for the compact case.

2. Notation and terminology

The symbol $\mathbb{R}$ stands for the real line with its natural topology. All spaces are assumed to be Tychonoff. Given spaces $X$ and $Y$ the set $C(X, Y)$ consists of continuous maps from $X$ to $Y$; we write $C(X)$ instead of $C(X, \mathbb{R})$. The expression $C_p(X, Y)$ denotes the set $C(X, Y)$ endowed with the pointwise convergence topology, i.e., $C_p(X, Y)$ (or $C_p(X)$ respectively) is $C(X, Y)$ (or $C(X)$ respectively) with the topology inherited from $Y^X (\mathbb{R}^X)$. We also let $C_p,0(X) = X$ and $C_p,n+1(X) = C_p(C_p,n(X))$ for any $n \in \omega$. A space $Z$ is said to have *countable tightness* (this is denoted by $t(Z) = \omega$) if, for any $A \subset Z$ and $z \in \overline{A}$ we have $z \in \overline{B}$ for some countable $B \subset A$.

3. Metacompactness in $C_p(X)$ for general spaces $X$

The most ambitious purpose would be to characterize metacompactness of $C_p(X)$ in terms of the space $X$. We do not formulate this as a question because
even for the Lindelöf property, its characterization in $C_p(X)$ seems to be a hopeless thing. The Souslin property of the spaces $C_p(X)$ implies that $C_p(X)$ is paracom pact if and only if it is Lindelöf (see Chapter 0, Section 1 of Arhangel’skiĭ [2]) so the following problem seems to be the most important one.

**Problem 1.** Does metacompactness of $C_p(X)$ imply that $C_p(X)$ is Lindelöf?

This question has long been a part of the folklore. The general theory of covering properties shows that if any closed discrete subspace of a metacompact $C_p(X)$ is countable then it is Lindelöf. It is a brilliant theorem of Reznichenko (see Theorem 1.5.12 of Arhangel’skiĭ [2]) that any normal $C_p(X)$ is collectionwise normal so metacompactness of $C_p(X)$ together with its normality implies that $C_p(X)$ is Lindelöf.

**Problem 2.** Suppose that $C_{p,n}(X)$ is metacompact for all $n \geq 1$. Must $C_p(X)$ be Lindelöf?

Maybe, the Lindelöf property is too much to ask from a metacompact $C_p(X)$. The following two problems present more humble expectations.

**Problem 3.** Suppose that $C_p(X)$ is metacompact. Must it be realcompact?

**Problem 4.** Let $X$ be a space such that $C_p(X)$ is metacompact. Is it true that the tightness of $X$ has to be countable?

If Problem 4 is answered positively then the answer to Problem 3 is also positive (see Chapter II, Section 4 of Arhangel’skiĭ [2]). Besides, if $C_p(X)$ is Lindelöf then $t(X) = \omega$ (see Asanov [3]) so Problem 4 asks whether it is possible to strengthen Asanov’s result.

**Problem 5.** Suppose that $C_p(X)$ is metacompact. Must $C_p(X) \times C_p(X)$ be metacompact?

This question is obligatory apart from reminding us the famous Arhangel’skiĭ problem (unanswered for several decades and published in many places, see e.g., Group of Problems C in Section 1 of Chapter 0 of Arhangel’skiĭ [2]) on whether the square of any Lindelöf $C_p(X)$ is Lindelöf. Maybe some weaker property of $C_p(X) \times C_p(X)$ can be derived from the Lindelöf property of $C_p(X)$ so it is worth to check for metacompactness.

**Problem 6.** Suppose that $C_p(X)$ is Lindelöf. Must $C_p(X) \times C_p(X)$ be metacompact?

The following two problems are related to the results of Tkachuk [6] on countable additivity of pseudocharacter, tightness, Čech-completeness and some other properties in $C_p(X)$. Outside of $C_p$-theory, it is easy to give examples of non-metacompact spaces which are countable unions of their closed metacompact subspaces.

**Problem 7.** Suppose that $C_p(X) = \bigcup_{n \in \omega} Y_n$ and every $Y_n$ is metacompact. Must $C_p(X)$ be metacompact?
Problem 8. Suppose that $C_p(X) = \bigcup_{n \in \omega} Y_n$ and every $Y_n$ is closed in $C_p(X)$ and metacompact. Must $C_p(X)$ be metacompact?

Since it is difficult to obtain any consequences of metacompactness in $C_p(X)$, we can suspect that it has some kind of universal presence in every function space. The following two problems formalize these suspicions.

Problem 9. Is it true that every $C_p(X)$ has a dense metacompact subspace?

Problem 10. Is it true that every space $X$ can be embedded in a space $Y$ such that $C_p(Y)$ is metacompact?

4. Metacompactness in $C_p(X)$ when $X$ is compact

If $X$ is compact then, to prove that $C_p(X)$ is Lindelöf, it suffices to establish some weaker properties of $C_p(X)$. For example, if $C_p(X)$ is normal then it is Lindelöf (see Theorem III.6.3 of Arhangel'skii [2]). Therefore it is mandatory to formulate the following clone of Problem 1.

Problem 11. Suppose that $X$ is compact and $C_p(X)$ is metacompact. Must $C_p(X)$ be Lindelöf?

The Lindelöf property of $C_p(X)$ has very strong consequences when $X$ is compact so we could expect that metacompactness of $C_p(X)$ implies some restrictions on $X$. The following question is the clone of Problem 4.

Problem 12. Let $X$ be a compact space such that $C_p(X)$ is metacompact. Is it true that $t(X) = \omega$?

The next two problems have positive answer if we replace metacompactness with the Lindelöf property. No counterexample exists for general spaces as well.

Problem 13. Let $X$ be a compact space such that $C_p(X)$ is metacompact. Is it true that $C_p(Y)$ is metacompact for any closed subspace $Y \subset X$?

Problem 14. Suppose that $X$ is compact and $C_p(X, [0, 1])$ is metacompact. Must $C_p(X)$ be metacompact?

Since preservation of topological properties in finite powers is often a key matter, we also have to present the clones of Problem 5 and Problem 6.

Problem 15. Suppose that $X$ is compact and the space $C_p(X)$ is metacompact. Must $C_p(X) \times C_p(X)$ be metacompact?

Problem 16. Suppose that $X$ is compact and $C_p(X)$ is Lindelöf. Must the space $C_p(X) \times C_p(X)$ be metacompact?

Every dyadic compact space of countable tightness is metrizable according to Theorem 3.1.1 of Arhangel'skii [1] so the Lindelöf property of $C_p(X)$ implies metrizability of any dyadic compact space $X$. The following question is again about what is left in $X$ if $C_p(X)$ is metacompact.
Problem 17. Suppose that $X$ is a dyadic compact space such that $C_p(X)$ is metacompact. Must $X$ be metrizable?

The space $\omega_1 + 1$ is a model for many matters concerned with countable tightness in compact spaces. It is known that $C_p(\omega_1 + 1)$ is very far from being Lindelöf; it does not even have a Lindelöf dense subspace (see Proposition IV.11.7 in Arhangel’skiǐ [2]); this easily implies that $\omega_1 + 1$ cannot be embedded in any $X$ such that $C_p(X)$ has a Lindelöf dense subspace.

On the other hand, Dow, Junnila and Pelant proved in [5] that for any compact $X$ of weight at most $\omega_1$, the space $C_p(X)$ is hereditarily metaLindelöf. Therefore we can expect some kind of metacompactness in $C_p(\omega_1 + 1)$. Our last three questions are intended to express these expectations.

Problem 18. Is the space $C_p(\omega_1 + 1)$ metacompact?

Problem 19. Is it true that $C_p(\omega_1 + 1)$ has a dense metacompact subspace?

Problem 20. Can the space $\omega_1 + 1$ be embedded in a space $Y$ such that $C_p(Y)$ is metacompact?

References


**Open problems in infinite-dimensional topology**

Taras Banakh, Robert Cauty and Michael Zarichnyi

**Introduction**

The development of Infinite-Dimensional Topology was greatly stimulated by three famous open problem lists: that of Geoghegan [58], West [74] and Dobrowolski, Mogilski [55]. We hope that the present list of problems will play a similar role for further development of Infinite-Dimensional Topology.

We expect that future progress will happen on the intersection of Infinite-Dimensional Topology with neighbor areas of mathematics: Dimension Theory, Descriptive Set Theory, Analysis, Theory of Retracts. According to this philosophy we formed the current list of problems. We tried to select problems whose solution would require creating new methods.

We shall restrict ourselves by separable and metrizable spaces. A *pair* is a pair \((X, Y)\) consisting of a space \(X\) and a subspace \(Y \subseteq X\). By \(\omega\) we denote the set of non-negative integers.

1. **Higher-dimensional descriptive set theory**

Many results and objects of infinite-dimensional topology have zero-dimensional counterparts usually considered in Descriptive Set Theory. As a rule, “zero-dimensional” results have simpler proofs compared to their higher-dimensional counterparts. Some zero-dimensional results are proved by essentially zero-dimensional methods (like those of infinite game theory) and it is an open question to which extent their higher-dimensional analogues are true. We start with two problems of this sort.

For a class \(C\) of spaces and a number \(n \in \omega \cup \{\infty\}\) consider the subclasses \(C[n] = \{C \in C : \dim C \leq n\}\) and \(C[\omega] = \bigcup_{n \in \omega} C[n]\). Following the tradition of Logic and Descriptive Set Theory, by \(\Sigma^1_1\) we denote the class of analytic spaces, i.e., metrizable spaces which are continuous images of \(\mathbb{N}^\omega\). Also \(\Pi^0_\alpha\) and \(\Sigma^0_\alpha\) stand for the multiplicative and additive classes of absolute Borel spaces corresponding to a countable ordinal \(\alpha\). In particular, \(\Pi^0_1\), \(\Pi^0_2\), and \(\Sigma^0_2\) are the classes of compact, Polish, and \(\sigma\)-compact spaces, respectively. In topology those classes usually are denoted by \(\mathcal{M}_0\), \(\mathcal{M}_1\), and \(\mathcal{A}_1\), respectively.

Following the infinite-dimensional tradition, we define a space \(X\) to be \(C\)-universal for a class \(C\) of spaces, if \(X\) contains a closed topological copy of each space \(C \in C\). According to a classical result of Descriptive Set Theory [62, 26.12], an analytic space \(X\) is \(\Pi^0_\xi[0]\)-universal for a countable ordinal \(\xi \geq 3\) if and only if \(X \notin \Sigma^0_\xi\). This observation implies that a space \(X\) is \(\Pi^0_\xi[0]\)-universal if and only if the product \(X \times Y\) is \(\Pi^0_\xi[0]\)-universal for some/any space \(Y \in \Sigma^0_\xi\). The philosophy of this result is that a space \(X\) is \(C\)-universal if \(X \times Y\) is \(C\)-universal for a relatively simple space \(Y\).
The following theorem proved in [25, 3.2.12] shows that in some cases this philosophy is realized also on the higher-dimensional level.

**Theorem.** Let $\mathcal{C} = \Pi_\xi^n$ where $n \in \omega \cup \{\infty\}$ and $\xi \geq 3$ is a countable ordinal. A space $X$ is $\mathcal{C}$-universal if and only if $X \times Y$ is $\mathcal{C}$-universal for some space $Y \in \Sigma_3^0$.

However we do not know if the condition $Y \in \Sigma_3^0$ can be replaced with a weaker condition $Y \in \Sigma_3^0$ (which means that $Y$ is an absolute $G_{\delta\sigma}$-space).

**Question 1.1.** Let $\mathcal{C} = \Pi_\xi^n$ where $n \in \omega \cup \{\infty\}$ and $\xi \geq 3$ be a countable ordinal. Is a space $X$ is $\mathcal{C}$-universal if $X \times Y$ is $\mathcal{C}$-universal for some space $Y \in \Sigma_3^0$?

As we already know the answer to this problem is affirmative for $n = 0$.

In fact, an affirmative answer to Question 1.1 would follow from the validity of the higher-dimensional version of the Separation Theorem of Louveau and Saint-Raymond [62, 28.18]. Its standard formulation says that two disjoint analytic sets $A, B$ in a Polish space $X$ cannot be separated by a $\Sigma_0^\xi$-set with $\xi \geq 3$ iff the pair $(A \cup B, A)$ is $(\Pi_0^0[0], \Pi_\xi^\xi)$-universal.

A pair $(X, Y)$ of spaces is defined to be $\mathcal{C}$-universal for a class of pairs $\mathcal{C}$ if for every pair $(A, B) \in \mathcal{C}$ there is a closed embedding $f: A \to X$ with $f^{-1}(Y) = B$. For classes $\mathcal{A}, \mathcal{B}$ of spaces by $(A, B)$ we denote the class of pairs $(A, B)$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We recall that $\Pi_1^\xi$ stands for the class of compacta.

The Separation Theorem of Louveau and Saint-Raymond implies that for every $\Pi_\xi^0[0]$-universal subspace $X$ of a space $Y \in \Sigma_3^\xi$ the pair $(Y, X)$ is $(\Pi_0^0[0], \Pi_\xi^\xi)$-universal. The philosophy of this result is clear: if a $\mathcal{C}$-universal space $X$ for a complex class $\mathcal{C}$ embeds into a “relatively simple” space $Y$, then the pair $(Y, X)$ is $(\Pi_0^0[0], \mathcal{C})$-universal. If the “relatively simple” means “$\sigma$-compact”, then the above zero-dimensional result has a higher-dimensional counterpart proved in [25, 3.1.2] (see also [16] and [50]).

**Theorem.** Let $n \in \omega \cup \{\infty\}$ and $\mathcal{C} \in \{\Pi_\xi^n, \Sigma_\xi^n : \xi \geq 3\}$. For every $\mathcal{C}[n]$-universal subspace $X$ of a space $Y \in \Sigma_3^n$ the pair $(Y, X)$ is $(\Pi_0^0[n], \mathcal{C})$-universal.

We do not know if $Y \in \Sigma_3^n$ in this theorem can be replaced with $Y \in \Sigma_3^0$.

**Question 1.2.** Let $n \in \omega \cup \{\infty\}$ and $\mathcal{C} = \Pi_\xi^n$ for a countable ordinal $\xi \geq 3$. Is it true that for each $\mathcal{C}[n]$-universal subspace $X$ of a space $Y \in \Sigma_3^\xi$ the pair $(Y, X)$ is $(\Pi_0^0[n], \mathcal{C})$-universal?

As we already know the answer to this question is affirmative for $n = 0$. Using Theorem 3.2.12 of [25] on preservation of $\mathcal{C}$-universality by perfect maps one can show that an affirmative answer to Question 1.2 implies that to Question 1.1.

Our third problem with a higher-dimensional descriptive flavor asks if higher-dimensional Borel complexity can be concentrated on sets of a smaller dimension. First let us make two simple observations: the Hilbert cube $[0, 1]^\omega$ is $\Pi_1^1$-universal while its pseudointerior $(0, 1)^\omega$ is $\Pi_2^0$-universal. In light of these observations one could suggest that for each $\xi \geq 1$ there is a one-dimensional space $X$ whose
countable power $X^\omega$ is $\Pi^0_\xi$-universal. But this is not true: no finite-dimensional space $X$ has $\Sigma^0_\xi$-universal countable power $X^\omega$ (see [17, 24, 42]). On the other hand, for every meager locally path-connected space $X$ the $(2n + 1)$-st power $X^{2n+1}$ is $\Sigma^0_\xi$-universal (which means that $X^{2n+1}$ contains a closed topological copy of each $n$-dimensional $\sigma$-compact space), see [18].

**Question 1.3.** Let $\mathcal{B} \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 1\}$ be a Borel class. Is there a one-dimensional space $X$ (in $\mathcal{B}$) with $\mathcal{B}[\omega]$-universal power $X^\omega$?

The answer to this problem is affirmative for the initial Borel classes $\mathcal{B} \in \{\Pi^0_1, \Pi^0_2, \Sigma^0_2\}$, see [18, 21, 36]. Moreover, for such a class $\mathcal{B}$ a space $X$ with $\mathcal{B}[n]$-universal power $X^{n+1}$ can be chosen as a suitable subspace of a dendrite with a dense set of end-points.

A related question concerns the universality in classes of compact spaces. It is well-known that the $n$-dimensional cube $[0,1]^n$ is not $\Pi^0_1$-universal. On the other hand, for any dendrite $D$ with dense set of end-points the power $D^{n+1}$ is $\Pi^0_1$-universal [36], and the product $D^{n+1} \times [0,1]^{2n}$ is $\Pi^0_1[2n]$-universal, see [28].

**Question 1.4.** Is $X \times [0,1]^{2n}$ $\Pi^0_1[2n]$-universal for any $\Pi^0_1[n]$-universal space $X$? Equivalently, is $\mu^n \times [0,1]^{2n}$ $\Pi^0_1[2n]$-universal (where $\mu^n$ denotes the $n$-dimensional Menger cube)?

For $n \leq 1$ the answer to this problem is negative. We expect that this is so for all $n$.

### 2. $Z_n$-sets and related questions

In this section we consider some problems related to $Z_n$-sets, where $n \in \omega \cup \{\infty\}$. By definition, a subset $A$ of a space $X$ is a $Z_n$-set in $X$ if $A$ is closed and the complement $X \setminus A$ is $n$-dense in $X$ in the sense that each map $f : [0,1]^n \to X$ can be uniformly approximated by maps into $X \setminus A$. In particular, $0$-density is equivalent to the usual density and a subset $A \subset X$ is a $Z_0$-set in $X$ if and only if it is closed and nowhere dense in $X$.

A set $A \subset X$ is a $\sigma Z_n$-set if $A$ is the countable union of $Z_n$-sets in $X$. A subset $A \subset X$ is called $n$-meager if $A \subset B$ for some $\sigma Z_n$-set $B$ in $X$. A space $X$ is a $\sigma Z_n$-space (or else $n$-meager) if $X$ is a $\sigma Z_n$-set (equivalently $n$-meager) in itself. In particular, a space is $0$-meager if and only if it is of the first Baire category.

According to a classical result of S. Banach [6], an analytic topological group either is complete or else is $0$-meager. It is natural to ask about the infinite version of this result. Namely, Question 4.4 in [55] asks if any incomplete Borel pre-Hilbert space is $\infty$-meager. The answer to this question turned out to be negative: the linear span($E$) of the Erdős set $E \subset \{(x_i) \in l_2 : \forall i \in \mathbb{Q}\}$ is meager but not $\infty$-meager, see [25, 5.5.19]. Moreover, span($E$) cannot be written as the countable union $\bigcup_{n \in \omega} Z_n$ where each set $Z_n$ is a $Z_n$-set in span($E$). On the other hand, for every $n \in \omega$, span($E$) can be written as the countable union of $Z_n$-sets, see [10].

**Question 2.1.** Is an (analytic) linear metric space $X$ a $\sigma Z_\infty$-space if $X$ can be written as the countable union $X = \bigcup_{n \in \omega} X_n$ where each set $X_n$ is a $Z_n$-set in $X$. 


By its spirit this problem is related to Selection Principles, a branch of Combinatorial Set Theory considered in the papers [69, 72].

Another feature of span(E) leads to the following problem, first posed in [7].

**Question 2.2.** Is every analytic non-complete linear metric space X a σZ_n-space for every n ∈ ω? Is this true if X is a linear subspace of ℓ^2 or R^ω?

With the help of the Multiplication Formula for σZ_n-spaces [27] or [20], the (second part of the) above problem can be reduced to the following one.

**Question 2.3.** Let X be a non-closed analytic linear subspace of the space L = ℓ^2 or L = R^ω. Can L be written as the direct sum L = L_1 ⊕ L_2 of two closed subspaces L_1, L_2 ⊂ L so that for every i ∈ {1, 2} the projection X_i of X onto L_i is a proper subspace of L_i?

Let us note that the zero-dimensional counterpart of this question has an affirmative solution: for each meager subset H ⊂ {0, 1}^ω there is a partition ω = A ∪ B of ω into two disjoint sets A, B such that the projections of H onto {0, 1}^A and {0, 1}^B are not surjective. This partition can be easily constructed by induction.

The following weaker problem related to Question 2.3 also is open.

**Question 2.4.** Let X be a non-closed analytic linear subspace in ℓ^2. Is there a closed infinite-dimensional linear subspace L ⊂ ℓ^2 such that X + L ≠ ℓ^2?

We recall that a space X is (strongly) countable-dimensional if X can be written as the countable union X = ∪_{n=1}^∞ X_n of (closed) finite-dimensional subspaces of X. The space span(E) is countable-dimensional but not strongly countable-dimensional, see [10].

**Question 2.5.** Is each (analytic) strongly countable-dimensional linear subspace of ℓ^2 ∞-meager? equivalently, 2-meager?

In light of this question it should be mentioned that each closed finite-dimensional subspace of the Hilbert space ℓ^2 is a Z_1-set. On the other hand, ℓ^2 contains closed zero-dimensional subsets failing to be Z_2-sets in ℓ^2. Yet, each finite-dimensional Z_2-set in ℓ^2 is a Z_∞-set in ℓ^2, see [64].

Let M_n be the σ-ideal consisting of n-meager subsets of the Hilbert cube Q. In particular, M_0 coincides with the ideal M of meager subsets of Q well studied in Set Theory. For each nontrivial ideal I of subsets of a set X we can study four cardinal characteristics:

- add(I) = min{|J| : J ⊂ I, ∪ J ∉ I};
- cov(I) = min{|J| : J ⊂ I, ∪ J = X};
- non(I) = min{|A| : A ⊂ X, A ∉ I};
- cof(I) = min{|C| : C ⊂ I, (∀A ∈ I)(∃C ∈ C)A ⊂ C}.

The cardinal characteristics of the ideal M_0 = M are calculated in various models of ZFC and can vary between κ_1 and the continuum c, see [73]. In [30] it is shown that cov(M_n) = cov(M) and non(M_n) = non(M) for every n ∈ ω ∪ {∞}. 

**Question 2.6.** Is \(\text{add}(M_n) = \text{add}(M)\) and \(\text{cof}(M_n) = \text{cof}(M)\) for every \(n \in \omega \cup \{\infty\}\)?

It is well-known that \(Z_\infty\)-sets in ANR-spaces can be characterized as closed sets with homotopy dense complement. A subset \(D\) of a space \(X\) is called *homotopy dense* if there is a homotopy \(h: X \times [0, 1] \to X\) such that \(h(x, 0) = x\) and \(h(x, t) \in D\) for all \((x, t) \in X \times (0, 1]\).

One of the problems from [55, 74] asked about finding an inner characterization of homotopy dense subspaces of \(s\)-manifolds. In [25, 1.3.2] (see also [8] and [52]) it was shown that such subspaces can be characterized with help of SDAP, the Toruńczyk’s Strong Discrete Approximation Property. This characterization allowed to apply powerful tools of the theory of Hilbert manifolds to studying spaces with SDAP.

**Question 2.7.** Find an inner characterization of homotopy dense subspaces of \(Q\)-manifolds.

The problem of the characterization of locally compact spaces homeomorphic to homotopy dense subsets of compact ANRs (or compact \(Q\)-manifolds) was considered in [46, 51].

It is known that each homotopy dense subspace \(X\) of a locally compact ANR-space has LCAP, the Locally Compact Approximation Property. The latter means that for every open cover \(U\) of \(X\) the identity map of \(X\) can be uniformly approximated by maps \(f: X \to X\) whose range \(f(X)\) has locally compact closure in \(X\).

**Question 2.8.** Is each space \(X\) with LCAP homeomorphic to a homotopy dense subspace of a locally compact ANR?

Let us note that LCAP appears as an important ingredient in many results of infinite-dimensional topology, see [9, 25].

**Question 2.9.** Let \(X\) be a convex set in a linear metric space. Has \(X\) LCAP? Has \(X\) LCAP if \(X\) is an absolute retract?

The answer to the latter question is affirmative if the completion of \(X\) is an absolute retract, see [25, 5.2.5].

### 3. The topology of convex sets and topological groups

One of the classical applications of infinite-dimensional topology is detecting the topological structure of convex sets in linear metric spaces. As a rule, convex sets are absolute retracts and have many other nice features facilitating applications of powerful methods of infinite-dimensional topology. Among such methods let us recall the theory of \(Q\)- and \(\ell^2\)-manifolds and the theory of absorbing and coabsorbing spaces. The principal notion unifying these theories is the notion of a strongly universal space.

A topological space \(X\) is defined to be *strongly \(C\)-universal* for a class \(C\) of spaces if for every cover \(U\), every space \(C \in C\) and a map \(f: C \to X\) whose restriction \(f|B: B \to X\) onto a closed subset \(B \subset C\) is a \(Z\)-embedding there is a
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Z-embedding \( \tilde{f} : C \to X \) which is \( \mathcal{U} \)-near to \( f \) and coincides with \( f \) on \( B \). A map \( f : C \to X \) is called a Z-embedding if it is a topological embedding and \( f(C) \) is a \( Z_\infty \)-set in \( X \). A topological space \( X \) is strongly universal if it is strongly \( Z(X) \)-universal for the class \( Z(X) \) of spaces homeomorphic to \( Z_\infty \)-sets of \( X \). Many natural spaces are strongly universal.

In [8, 13, 15, 25, 37, 38, 41, 45, 48, 54] many results on the strong universality of convex sets in linear metric space were obtained. Nonetheless the following problem still is open.

**Question 3.1.** Let \( X \) be an infinite-dimensional closed convex set in a locally convex linear metric space \( L \). Is \( X \) strongly universal?

The answer is not known even for the case when \( X \) is a pre-Hilbert space. A bit weaker question also is open.

**Question 3.2.** Let \( X \) be an infinite-dimensional closed convex set in a locally convex linear metric space \( L \). Is \( X \) strongly \( Z_{tb}(X) \)-universal for the class of spaces homeomorphic to totally bounded \( Z_\infty \)-subsets of \( X \)?

The answer to this problem is affirmative if \( X \) has an almost internal point \( x_0 \in X \) (the latter means that the set \( \{x \in X : (\exists z \in X)(\exists t \in (0, 1)) x_0 = tx + (1 - t)z\} \) is dense in \( X \)), see [13].

Strong universality enters as one of important ingredients into the definition of a (co)absorbing space. A topological space \( X \) is called absorbing (resp. coabsorbing) if \( X \) is an \( \infty \)-meager (resp. \( \infty \)-comeager) strongly universal ANR with SDAP. We recall that a space is \( n \)-meager where \( n \in \omega \cup \{\infty\} \) if it is a \( \sigma Z_n \)-set in itself. A space \( X \) is defined to be \( n \)-comeager if \( X \) contains an absolute \( G_\delta \)-subset \( G \) with \( n \)-meager complement \( X \setminus G \) in \( X \). In particular an analytic space is \( 0 \)-comeager if and only if it is Baire.

**Question 3.3.** Let \( X \) be a closed convex subset of a locally convex linear metric space. Assume that \( X \) is \( 0 \)-comeager. Is it \( \infty \)-comeager? Is \( X \) \( n \)-comeager for all \( n \in \omega \)?

**Question 3.4.** Assume that \( X \in AR \) is an \( \infty \)-(co)meager closed convex set in a linear metric space. Is \( X \) a (co)absorbing space?

The principal result of the theory of (co)absorbing spaces is the Uniqueness Theorem [25, §1.6] asserting that two (co)absorbing spaces \( X, Y \) are homeomorphic if and only if \( X, Y \) are homotopically equivalent and \( Z(X) = Z(Y) \). This fact helps to establish the topological structure of many infinite-dimensional (co)absorbing spaces appearing in nature, see [25, 41, 68].

In particular, in [25] it was shown that a closed convex subset \( X \) of a locally convex linear metric space is \( \Pi_\xi^0 \)-(co)absorbing for \( \xi \geq 2 \) if and only if \( X \) is a \( \Pi_\xi^0 \)-universal \( \infty \)-(co)meager space and \( X \in \Pi_\xi^0 \). The same result is true for additive Borel classes \( \Sigma_\xi^0 \) with \( \xi \geq 3 \). Surprisingly, but for the class \( \Pi_1^0 \) of compacta we still have an open question.
Question 3.5. Let $X$ be a $\Pi^0_1$-universal convex (closed $\sigma$-compact) subset of $\ell^2$. Is $X$ strongly $\Pi^0_1$-universal?

This question has an affirmative answer if $X$ contains an almost internal point. A similar situation holds for metrizable topological groups. It is shown in [25, 4.2.3] that a topological group $G$ is a $\Pi^0_1$-absorbing space for $\xi \geq 2$ if and only if $G \in \Pi^0_1$ is a $\Pi^0_1$-universal ANR.

Question 3.6. Let $G \in \text{ANR}$ be a $\Pi^0_1$-universal $\sigma$-compact metrizable group. Is $G$ an $\Pi^0_1$-absorbing space?

A similar question for the class $\Pi^0_1[\omega]$ of finite-dimensional compacta is also open, see [55, 5.7].

Question 3.7. Let $G$ be an infinite-dimensional $\sigma$-compact strongly countable-dimensional locally contractible group (containing a topological copy of each finite-dimensional compactum). Is $G$ a $\Pi^0_1[\omega]$-absorbing space? Equivalently, is $G$ an $\ell^2_{\gamma}$-manifold?

In fact, the method of absorbing sets works not only for spaces and pairs but also for order-preserving systems $(X_\gamma)_{\gamma \in \Gamma}$ of topological spaces, indexed by a partially ordered set $\Gamma$ with largest element $\max \Gamma$. The order-preserving property of $(X_\gamma)$ means that $X_\gamma \subset X_{\gamma'}$ for any elements $\gamma \leq \gamma'$ in $\Gamma$. So each $X_\gamma$ is a subspace of $X_{\max \Gamma}$. Such systems $(X_\gamma)$ are called $\Gamma$-systems. For a $\Gamma$-system $X = (X_\gamma)_{\gamma \in \Gamma}$, a subset $F \subset X_{\max \Gamma}$, and a map $f: Y \rightarrow X$ we let $F \cap X = (F \cap X_\gamma)_{\gamma \in \Gamma}$ and $f^{-1}(X) = (f^{-1}(X_\gamma))_{\gamma \in \Gamma}$.

The notion of strong universality extends to $\Gamma$-systems as follows: A $\Gamma$-system $X = (X_\gamma)_{\gamma \in \Gamma}$ is called strongly $\mathcal{C}$-universal for a class $\mathcal{C}$ of $\Gamma$-systems if given: an open cover $U$ of $X_{\max \Gamma}$, a $\Gamma$-system $A = (A_\gamma)_{\gamma \in \Gamma} \in \mathcal{C}$, a closed subset $F \subset A_{\max \Gamma}$ and a map $f: A_{\max \Gamma} \rightarrow X_{\max \Gamma}$ such that $f|F$ is a $Z$-embedding with $F \cap f^{-1}(X) = F \cap A$, there is a $Z$-embedding $\tilde{f}: A_{\max \Gamma} \rightarrow X_{\max \Gamma}$ such that $\tilde{f}$ is $U$-near to $f$, $\tilde{f}|F = f$ and $\tilde{f}^{-1}(X) = A$.

A system $X$ is called $\mathcal{C}$-absorbing in $E$ if $X$ is strongly $\mathcal{C}$-universal and $X_{\max \Gamma} = \bigcup_{n \in \omega} Z_n$ where each $Z_n$ is a $Z_\infty$-set in $X_{\max \Gamma}$ and $Z_n \cap X \in \mathcal{C}$. For more information on absorbing systems, see [5].

Given a class $\mathcal{C}$ of $\Gamma$-systems and a non-negative integer number $n$ consider the subclass

$$\mathcal{C}[n] = \{X \in \mathcal{C} : \dim(X_{\max \Gamma}) \leq n\}.$$ 

The following question is related to the results on existence of absorbing sets for $n$-dimensional Borel classes [76].

Question 3.8. For which classes $\mathcal{C}$ of $\Gamma$-systems does the existence of a $\mathcal{C}$-absorbing $\Gamma$-system imply the existence of a $\mathcal{C}[n]$-absorbing $\Gamma$-system for every $n \in \omega$?

One can formulate this question also for other types of dimensions, in particular, for extension dimension introduced by Dranishnikov [57].
4. Topological characterization of particular infinite-dimensional spaces

The theory of (co)absorbing spaces is applicable for spaces which are either \( \infty \)-meager or \( \infty \)-comeager. However some natural strongly universal spaces do not fall into either of these two categories. One of such spaces is span\((E)\), the linear hull of the Erdős set in \( \ell^2 \) which is a meager strongly universal AR with SDAP that fails to be \( \infty \)-meager, see [10, 53].

**Question 4.1.** Give a topological characterization of \( \text{span}(E) \). Is \( \text{span}(E) \) homeomorphic to the linear hull \( \text{span}(Q^\omega) \) of \( Q^\omega \) in \( \mathbb{R}^\omega \)? to the linear hull \( \text{span}(E_p) \) of the Erdős set \( E_p = \{ (x_i) \in \ell^p : (x_i) \in Q^\omega \text{ and } \lim_{i \to \infty} x_i = 0 \} \) in the Banach space \( \ell^p \), \( 1 \leq p \leq \infty \)?

Another problem of this sort concerns the countable products \( X^\omega \) of finite-dimensional meager absolute retracts \( X \). Using [25, 4.1.2] one can show that the countable product of such a space \( X \) is a strongly universal AR with SDAP. The space \( X^\omega \) is a \( n \)-meager for all \( n \in \omega \) but unfortunately is not \( \infty \)-meager.

**Question 4.2.** Let \( X, Y \) be finite-dimensional \( \sigma \)-compact absolute retracts of the first Baire category. Are \( X^\omega \) and \( Y^\omega \) homeomorphic? (Applying [18] one can show that each of the spaces \( X^\omega, Y^\omega \) admits a closed embedding into the other space.)

Our third pathological (though natural) example is the hyperspace \( \exp_H(Q_I) \) of closed subsets of the space of rationals \( Q_I = [0,1] \cap \mathbb{Q} \) on the interval, endowed with the Hausdorff metric. This space has many interesting features similar to those of span \( E \): \( \exp_H(Q_I) \) is \( n \)-meager for all \( n \in \omega \) but fails to be \( \infty \)-meager; \( \exp_H(Q_I) \) is homeomorphic to its square and belongs to the Borel class \( \Pi_3^0 \)-universal but fails to be \( \Pi_3^0 \)-universal, see [23].

**Question 4.3.** Give a topological characterization of the space \( \exp_H(Q_I) \). In particular, are the spaces \( \exp_H(Q_I) \) and \( \exp_H(Q_I \times Q_I) \) homeomorphic?

The three preceding examples were meager but not \( \infty \)-meager. Because of that they cannot be treated by the theory of absorbing spaces. Our other two problems concern spaces that are \( 0 \)-comeager but not \( \infty \)-comeager and hence cannot be treated by the theory of coabsorbing spaces. These spaces are defined with help of the operation of weak product

\[
W(X,Y) = \{(x_i) \in X^\omega : (\exists n \in \omega)(\forall i \geq n) \ x_i \in Y\}
\]

where \( Y \) is a subspace of \( X \). The classical space of the form \( W(X,Y) \) is the Nagata space \( N = W(\mathbb{R}, \mathbb{P}) \) well-known in Dimension Theory as a universal space in the class of countable-dimensional (absolute \( G_{\delta_\sigma} \))-spaces. Here \( \mathbb{P} = \mathbb{R} \setminus \mathbb{Q} \) stands for the space of irrationals. The countable product \( \mathbb{P}^\omega \) is a dense \( G_\delta \)-set in \( W(\mathbb{R}, \mathbb{P}) \). Nonetheless, \( W(\mathbb{R}, \mathbb{P}) \) contains no \( \infty \)-dense absolute \( G_\delta \)-set (because \( W(\mathbb{R}, \mathbb{P}) \) is countable-dimensional) and thus \( W(\mathbb{R}, \mathbb{P}) \) is not a coabsorbing space (but is strongly universal and has SDAP).
Question 4.4. Give a topological characterization of the Nagata space $N = W(\mathbb{R}, \mathbb{P})$.

To pose a (possibly) more tractable question, let us note that $N$ is homeomorphic to $W(N, \mathbb{P}^\omega)$ (by a coordinate-permuting homeomorphism).

Question 4.5. Is $N = W(\mathbb{R}, \mathbb{P})$ homeomorphic to $W(N, (\mathbb{P} \setminus \{\sqrt{2}\})^\omega)$?

Next, we shall ask about the characterization of the pair $(I^\omega, \mathbb{P}^\omega_I)$ where $\mathbb{P}^\omega_I = I \cap \mathbb{P}$ is the set of irrational numbers on the interval $I = [0, 1]$. Topological characterizations of the Hilbert cube $I^\omega$ and irrational numbers $\mathbb{P}^\omega_I$ are well-known.

Question 4.6. Give a topological characterization of the pair $(I^\omega, \mathbb{P}^\omega_I)$. In particular, is $(I^\omega, \mathbb{P}^\omega_I)$ homeomorphic to $(I^\omega, G)$ for every dense zero-dimensional $G_\delta$-subset $G \subset I^\omega$ with homotopy dense complement in $I^\omega$?

A similar question concerns also the pair $(I^\omega, \mathbb{Q}^\omega_I)$ where $\mathbb{Q}^\omega_I = I \cap \mathbb{Q}$. Since $\mathbb{Q}^\omega_I$ is not $\infty$-meager in $I^\omega$, this pair can not be treated by the theory of absorbing pairs, see [25].

Question 4.7. Give a topological characterization of the pair $(I^\omega, \mathbb{Q}^\omega_I)$.

5. Problems on ANRs

One of the principal problems on ANRs from the preceding two lists [58, 74], the classical Borsuk’s Problem on the AR-property of linear metric spaces, has been resolved in negative by R. Cauty in [40] who constructed a $\sigma$-compact linear metric space that fails to be an absolute retract. However, the “compact” version of Borsuk’s problem still is open.

Question 5.1. Let $C$ be a compact convex set in a linear metric space. Is $C$ an absolute retract?

There are also many other natural spaces whose ANR-property is not established. Some of them are known to be divisible by the Hilbert space $\ell^2$ in the sense that they are homeomorphic to the product with $\ell^2$ (and hence are $\ell^2$-manifolds if and only if they are ANRs). A classical example of this sort is the homeomorphism group of an $n$-manifolds for $n \geq 2$, see [74, HS4].

Another example is the space $H_B$ of Brouwer homeomorphisms of the plane, endowed with the compact-open topology. A homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a Brouwer homeomorphism if $h$ preserves the orientation and has no fixed point.

Question 5.2. Is the space $H_B$ an ANR?

It is known that $H_B$ is locally contractible [34], is homotopically equivalent to the circle $S^1$ and is divisible by $\ell^2$ [67]. So, $H_B$ is homeomorphic to $S^1 \times \ell^2$ if and only if $H_B$ is an ANR.

In spite of the existence of a linear metric space failing to be an AR, R. Cauty proved that each convex subset of a linear metric space is an algebraic ANR. (algebraic ANRs are defined with help of a homological counterpart of the Lefschetz
condition, see [44]). This follows from an even more general fact asserting that each metrizable locally equiconnected space is an algebraic ANR, see [44].

We recall that a topological space $X$ is \textit{locally equiconnected} if there are open neighborhoods $U \subset X \times X$ of the diagonal and a continuous function $\lambda: U \times [0,1] \to X$ such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$ and $\lambda(x,x,t) = x$ for every $(x,y,t) \in U \times [0,1]$. If $U = X \times X$, then $X$ is called \textit{equiconnected}. It is easy to see that each (locally) contractible topological group $G$ is (locally) equiconnected and so is any retract of $G$. We do not know if the converse is true.

\textbf{Question 5.3.} Let $X$ be a (locally) equiconnected metrizable space. Is $X$ a (neighborhood) retract of a contractible metrizable topological group?

It should be mentioned that this question has an affirmative answer for compact $X$, see [39]. The proof of this particular case exploits the fact that each metrizable equiconnected space $X$ admits a Mal’tsev operation (which is a continuous map $\mu: X^3 \to X$ such that $\mu(x,x,y) = \mu(y,x,x) = y$ for all $x, y \in X$). Due to Sipacheva [71] we know that each compact space $X$ admitting a Mal’tsev operation is a retract of the free topological group $F(X)$ over $X$. Therefore, each equiconnected compact metrizable space $X$ has a Mal’tsev operation and hence is a retract of the free topological group $F(X)$. Moreover, it can be shown that the connected component of $F(X)$ containing $X$ is contractible. Now it is easy to select a metrizable group topology $\tau$ on $F(X)$ inducing the original topology on $X$ and such that $X$ still is a retract of $(F(X), \tau)$ and the component of $(F(X), \tau)$ containing $X$ is contractible. This resolves the “compact” version of Question 5.3. The non-compact version of this problem is related to the following question (discussed also in [63]):

\textbf{Question 5.4.} Is a metrizable space admitting a Mal’tsev operation a retract of a metrizable topological group?

By definition, an \textit{n-mean} on a topological space $X$ is a continuous map $m: X^n \to X$ such that $m(x,\ldots,x) = x$ for all $x \in X$ and $m(x_{\sigma(1)},\ldots,x_{\sigma(n)}) = m(x_1,\ldots,x_n)$ for any vector $(x_1,\ldots,x_n) \in X^n$ and any permutation $\sigma$ of $\{1,\ldots,n\}$. Let us note that each convex subset $C$ of a linear topological space admits an $n$-means and so does any retract of $C$.

\textbf{Question 5.5.} Let $n \geq 2$. Is there a metrizable equiconnected compact space $X$ admitting no $n$-mean?

If such a compact space $X$ exists then it is a retract of a contractible group but fails to be a retract of a convex subset of a linear topological space.

For a compact space $X$ let $L(X)$ be the free topological linear space over $X$ and $P(X)$ be the convex hull of $X$ in $L(X)$ (it can be shown that $P(X)$ is a free convex set over $X$). Let $(U(X), \lambda_X)$ be the free equiconnected space over $X$ (where $\lambda: U(X) \times U(X) \times [0,1] \to U(X)$ is the equiconnected map of $U(X)$), see [43]. Let $T_v(X)$ be the family of metrizable linear topologies on $L(X)$ inducing the original topology on $X$. (The family $T_v(X)$ was essentially used in [40] for constructing the example of a linear metric space failing to be an AR). Let $T_v(X) = \{\tau | P(X) :
\( \tau \in T_v(X) \) be the family consisting of the restrictions of the topologies \( \tau \in T_v(X) \) onto \( P(X) \), and \( T_u(X) \) be the family of metrizable topologies on \( U(X) \) which induce the initial topology on \( X \) and preserve the continuity of the equiconnected map \( \lambda_X : U(X) \times U(X) \times [0,1] \to U(X) \).

It is interesting to study the classes \( A_v \), \( A_c \), \( A_u \) of metric compacta \( X \) such that the spaces \( (L(X), \tau) ((P(X), \tau), (U(X), \tau)) \) are absolute retracts for all topologies \( \tau \) in \( T_v(X) \) \( (T_c(X), T_u(X)) \). It is known that \( A_u \subset A_c \subset A_v \), see [43].

Question 5.6. Is it true that \( A_u = A_c = A_v \)?

The class \( A_u \) contains all metrizable compact ANRs and all metrizable compact \( C \)-spaces, see [43].

Question 5.7. Is it true that each weakly infinite-dimensional compact metrizable space belongs to \( A_v \)? to \( A_u \)?

In light of this question let us mention that there is a strongly infinite-dimensional compact space \( D \) of finite cohomological dimension with \( D \notin A_v \), see [40]. In fact, the free linear space \( L(D) \) over \( D \), endowed with a suitable metrizable topology, gives the mentioned example of a linear metric space which is not an AR.

According to [43] the class \( A_c \) is closed with respect to countable products. We do not know if the same is true for the class \( A_u \).

Question 5.8. Is the class \( A_u \) closed with respect to countable products?

6. Infinite-dimensional problems from Banach space theory

In this section we survey some open problems lying in the intersection of infinite-dimensional topology and the theory of Banach spaces. Our principal object is the unit ball \( B_X = \{ x \in X : \|x\| \leq 1 \} \) of a Banach space \( X \), endowed with the weak topology. It is well-known that the weak ball \( B_X \) is metrizable (and separable) if and only if the Banach space \( X \) has separable dual. So, till the end of this section by a “Banach space” we understand an infinite-dimensional Banach space with separable dual. In [12] the following general problem was addressed:

**Question 6.1.** Investigate the interplay between geometric properties of a Banach space \( X \) and topological properties of its weak unit ball \( B_X \). Find conditions under which two Banach spaces have homeomorphic weak unit balls.

It turns out that answers to these questions depend on (1) the class \( W(X) \) of topological spaces homeomorphic to closed bounded subsets of a Banach space \( X \) endowed with the weak topology, and (2) [in case of complex \( W(X) \)] on properties of the norm of \( X \). Let us remark that \( W(X) \) coincides with the class \( F_0(B_X) \) of topological spaces homeomorphic to closed subsets of the weak unit ball \( B_X \) of \( X \).

In [12] it was observed that the class \( W(X) \) is not too large: it lies in the class \( \Pi_3^0 \) of absolute \( F_{\sigma\delta} \)-sets. For reflexive infinite-dimensional Banach spaces \( X \) the class \( W(X) \) coincides with the class \( \Pi_1^0 \) of compacta. On the other hand, for
the Banach space $X = c_0$ the class $W(X)$ is the largest possible and coincides with the Borel class $\Pi^0_3$. An intermediate case $W(X) = \Pi^0_2$ happens if and only if $X$ is a non-reflexive Banach space with PCP, the Point Continuity Property (which means that for each bounded weakly closed subset $B \subset X$ the identity map $(B, \text{weak}) \to B$ has a continuity point).

For Banach spaces with PCP the weak unit ball $B_X$ is homeomorphic either to the Hilbert cube $Q$ (if $X$ is reflexive) or to the pseudointerior $s = (0, 1)^\omega$ of $Q$ (if $X$ is not reflexive). In two latter cases, the topology of $B_X$ does not depend on the particular choice of an equivalent norm on $X$. In this case we say that the Banach space $X$ has BIP, the Ball Invariance Property. More precisely, $X$ has BIP if the weak unit ball $B_X$ of $X$ is homeomorphic to the weak unit ball $B_Y$ of any Banach space $Y$, isomorphic to $X$. It is known [12] that PCP implies BIP and BIP implies CPCP, the Convex Point Continuity Property, which means that each closed convex bounded subset of the Banach space has a point at which the norm topology coincides with the weak topology. It is known that the properties PCP and CPCP are different: the Banach space $B_\infty$ constructed in [59] has CPCP but not PCP.

1273? **Question 6.2.** Is BIP equivalent to PCP? To CPCP? Has the Banach space $B_\infty$ BIP?

In fact, the geometric properties PCP, BIP, and CPCP of a Banach space $X$ can be characterized via topological properties of the weak unit ball $B_X$: $X$ has PCP (resp. CPCP, BIP) if and only if $B_X$ is Polish ($0$-comeager, $\infty$-comeager).

The norm of a Banach space $X$ will be called $n$-(co)meager if the respective weak unit ball $B_X$ is $n$-(co)meager. Let us remark that each Kadec norm is $\infty$-comeager (since the unit sphere is an $\infty$-dense absolute $G_\delta$-subset in the weak unit ball). It is well-known that each separable Banach space admits an equivalent Kadec (and hence $\infty$-comeager) norm.

1274? **Question 6.3.** Give a geometric characterization of Banach spaces admitting an equivalent $\infty$-meager norm.

For $n$-meager norms with $n \in \omega$ the answer is known: a Banach space $X$ admits an equivalent $n$-meager norm if and only if $X$ fails to have CPCP. On the other hand, a Banach space $X$ has an equivalent $\infty$-meager norm if $X$ fails to be strongly regular, see [12]. We recall that a Banach space $X$ is called strongly regular if for every $\varepsilon > 0$ and every nonempty convex bounded subset $C \subset X$ there exist nonempty relatively weak-open subsets $U_1, \ldots, U_n \subset C$ such that the norm diameter of $\frac{1}{n} \sum_{i=1}^{n} U_i$ is less than $\varepsilon$. An example of a strongly regular Banach space $S, T_\infty$ failing to have CPCP was constructed in [60]. This space has an equivalent norm which is $n$-meager for every $n \in \omega$, see [12].

1275? **Question 6.4.** Is there a strongly regular Banach space admitting an equivalent $\infty$-meager norm? Has the space $S, T_\infty$ an equivalent $\infty$-meager norm?

If a Banach space admits a $0$-meager norm (equivalently, $X$ fails to have CPCP), then the class $W(X)$ contains all finite-dimensional absolute $F_{n,\delta}$-spaces.
If, moreover, the norm of a Banach space $X$ is $\infty$-meager, then $W(X) = \Pi_3^0$ and the weak unit ball $B_X$ is homeomorphic to the weak unit ball of the Banach space $c_0$ endowed with the standard sup-norm. We do not know if the Banach space $S_*T_\infty$ has an equivalent $\infty$-meager norm, but we know that $W(S_*T_\infty) = \Pi_3^0$ and the weak unit ball of $S_*T_\infty$ endowed with a Kadec norm is homeomorphic to the weak unit ball of $c_0$ endowed with a Kadec norm. The space $S_*T_\infty$ is an example of a strongly regular space with $W(S_*T_\infty) = \Pi_3^0$. However, $S_*T_\infty$ fails to have CPCP.

**Question 6.5.** Is there a Banach space $X$ with $W(X) = \Pi_3^0$ admitting no $\infty$-meager norm? having CPCP?

In light of this question it should be mentioned that each Banach space with PCP has $W(X) = \Pi_i^0$ for $i \in \{1, 2\}$. Also a Banach space with CPCP admits no 0-meager norm. It is known that the Banach space $c_0$ contains no conjugate subspaces and has $W(c_0) = \Pi_3^0$.

**Question 6.6.** Suppose $X$ is a Banach space with separable dual, containing no subspace isomorphic to a dual space. Is $W(X) = \Pi_3^0$?

For a Banach space $X$ with an $\infty$-meager norm the weak unit ball is an absorbing space (in fact, a $\Pi_3^0$-absorbing space).

Similarly, for a Banach space with $\infty$-comeager norm the weak unit ball $B_X$ is a coabsorbing space and its topology is completely determined by the class $W(X)$. The same concerns the topology of the pair $(B_{X^*}^*, B_X)$, where $B_{X^*}^*$ is the unit ball in the second dual Banach space $X^{**}$, endowed with the $*$-weak topology. The topology of this pair is completely determined by the class $W(X^{**}, X)$ of pairs $(K, C)$ homeomorphic to pairs of the form $(B, B \cap X)$ where $B \subseteq X^{**}$ is $w^*$-closed bounded subset of the second dual space $(X^{**}, weak^*)$.

More precisely, we have the following classification theorem of Cantor–Bernstein type proved in [12].

**Theorem** (Classification Theorem). Let $X, Y$ be Banach spaces with separable dual and $\infty$-comeager norms.

1. The weak unit balls $B_X$ and $B_Y$ are homeomorphic if and only if $W(X) = W(Y)$.
2. The pairs $(B_{X^*}^*, B_X)$ and $(B_{Y^*}^*, B_Y)$ are homeomorphic if and only if $W(X^{**}, X) = W(Y^{**}, Y)$.

It is clear that the topological equivalence of the pairs $(B_{X^*}^*, B_X)$ and $(B_{Y^*}^*, B_Y)$ implies the topological equivalence of the weak unit balls $B_X$ and $B_Y$. We do not know if the converse is also true.

**Question 6.7.** Assume that $X, Y$ are Banach spaces with homeomorphic weak unit balls $B_X$ and $B_Y$. Are the pairs $(B_{X^*}^*, B_X)$ and $(B_{Y^*}^*, B_Y)$ homeomorphic?

The answer to this question is affirmative provided $W(X) = W(Y) = \Pi_\xi^0$ for some $\xi \in \{1, 2, 3\}$.
The classification Theorem suggests introducing the partially ordered set
\[ \mathcal{W}_\infty = \{ \mathcal{W}(X) : X \text{ is an infinite-dimensional Banach space with separable dual} \} \]
inducing the following preorder of the family of Banach spaces: \( X \preceq \mathcal{W} Y \) if \( \mathcal{W}(X) \subset \mathcal{W}(Y) \) (equivalently, if the weak unit ball of \( X \) admits a closed embedding into the weak unit ball of \( Y \)).

Since each separable Banach space is isomorphic to a subspace of \( C[0,1] \), the set \( \mathcal{W}_\infty \) contains at most continuum elements. Note that the set \( \mathcal{W}_\infty \) is partially ordered by the natural inclusion relation.

It is easy to see that the poset \( \mathcal{W}_\infty \) has the smallest and largest elements: \( \Pi^0_0 = \mathcal{W}(\ell^2) \) and \( \Pi^0_3 = \mathcal{W}(c_0) \) corresponding to classes \( \mathcal{W}(X) \) of the Hilbert space \( \ell^2 \) and the Banach space \( c_0 \). Also it is known that the class \( \Pi^0_2 = \mathcal{W}(J) \) where \( J \) is the James quasireflexive space is a unique immediate successor of \( \Pi^0_1 \). For some time there was a conjecture that \( \mathcal{W}_\infty \) consists just of these three elements: \( \Pi^0_0, \Pi^0_2, \) and \( \Pi^0_3 \). However it was discovered in [12] that for the Banach space \( B_\infty \) (distinguishing the properties PCP and CPCP) the class \( \mathcal{W}(B_\infty) \) is intermediate between \( \mathcal{W}(J) \) and \( \mathcal{W}(c_0) \). So the poset \( \mathcal{W}_\infty \) appeared to be richer than expected.

**Question 6.8.** Investigate the ordered set \( \mathcal{W}_\infty \). In particular, is it infinite? Is it linearly ordered?

The pathological class \( \mathcal{W}(B_\infty) \) contains the class \( \Pi^0_3[0] \) of all zero-dimensional absolute \( F_{\sigma\delta} \)-spaces but not the class \( \Pi^0_3[1] \), see [12]. This suggests the following (probably difficult)

**Question 6.9.** Let \( n \in \omega \). Is there a Banach space \( X \) such that \( \Pi^0_3[n] \subset \mathcal{W}(X) \) but \( \Pi^0_3[n+1] \not\subset \mathcal{W}(X) \)? (Such a space \( X \) if exists has CPCP but not PCP.)

**Question 6.10.** Is there a Banach space \( X \) such that \( \Pi^0_3[\omega] \subset \mathcal{W}(X) \) but \( \Pi^0_3 \not\subset \mathcal{W}(X) \)? (Such a space \( X \) if exists is strongly regular but fails to have PCP.)

In fact, the pathological space \( B_\infty \) is one of the spaces \( J_*T_{\infty,n}, n \geq 0, \) constructed in [60].

**Question 6.11.** Is \( \mathcal{W}(J_*T_{\infty,n}) \neq \mathcal{W}(J_*T_{\infty,m}) \) for \( n \neq m \)?

Another two questions concern the influence of operations over Banach spaces on the classes \( \mathcal{W}(X) \).

**Question 6.12.** Is \( \mathcal{W}(X \oplus Y) = \max\{\mathcal{W}(X), \mathcal{W}(Y)\} \) for infinite-dimensional Banach spaces \( X \) and \( Y \) with separable duals?

**Question 6.13.** Let \( X \) be an infinite-dimensional Banach space. Is \( \mathcal{W}(X \oplus X) = \mathcal{W}(X) \)? Is \( \mathcal{W}(X \oplus \mathbb{R}) = \mathcal{W}(X) \)?

Note that an infinite-dimensional Banach space \( X \) with separable dual need not be isomorphic to \( X \oplus X \) or \( X \oplus \mathbb{R} \), see [61].

In Figure 1 we collect all known information on the relationship between geometric properties of a Banach space \( X \) with separable dual, topological properties of the weak unit ball \( B_X \) and properties of the class \( \mathcal{W}(X) \). In the first line of the
diagram FD means “finite-dimensional”, R “reflexive”, and SR “strongly regular”. The second line of the diagram means that every equivalent weak unit ball \( B_X \) of \( X \) has the corresponding property; the third line means that the class \( \mathcal{W}(X) \) does not contain the corresponding class of absolute \( F_{\sigma\delta} \)-spaces. The slashed and curved arrows indicate that the corresponding implication is false (with a counterexample written near the slashed arrow).

Finally we ask some questions on the topological structure of operator images. By an operator image we understand an infinite-dimensional normed space of the form \( T \{X\} \) for a suitable linear continuous operator \( T : X \rightarrow Y \) between separable Banach spaces. In [22] it was shown that each operator image belongs to a Borel class \( \Pi_0^{\alpha+2} \), \( \Sigma_0^{\alpha+1} \), \( D_2(\Pi_0^{\alpha+1}) \) \( \setminus (\Pi_0^{\alpha+1} \cup \Sigma_0^{\alpha+1}) \) for a suitable ordinal \( \alpha \) (here \( D_2(\Pi_0^{\alpha+1}) \) is the class consisting of differences \( X \setminus Y \) with \( X, Y \in \Pi_0^{\alpha+1} \)) and each such a Borel class contains an operator image. Moreover, up to a homeomorphism each class \( \Pi_0^{\alpha+2}, \Sigma_0^{\alpha+1}, D_2(\Pi_0^{\alpha+1}) \) contains exactly one operator image. On the other hand, the class \( \Pi_0^{\alpha+2} \setminus \Sigma_0^{\alpha+1} \) contains at least two topologically distinct operator images, see [11, 22].

**Question 6.14.** Does the class \( \Sigma_0^{3} \setminus \Pi_3^{0} \) contain two topologically distinct operator images? The same question for other Borel classes.

The image \( T : X \rightarrow Y \) of a Banach space under a compact operator \( T \) always is an absorbing space, see [22]. Moreover, for every countable ordinal \( \alpha \geq 1 \) the multiplicative Borel class \( \Pi_0^{\alpha+2} \) contains an operator image which is a \( \Pi_0^{\alpha+2} \)-absorbing space. We do not know if the same is true for the additive Borel classes.

**Question 6.15.** Is there an operator image which is a \( \Sigma_0^{3} \)-absorbing space? a \( \Sigma_{\xi+1}^{3} \)-absorbing space with \( \xi \geq 1 \)?

For \( \xi = 1 \) the answer is affirmative: the image \( T : X \rightarrow Y \) of any reflexive Banach space under a compact bijective operator \( T : T \rightarrow Y \) is \( \Sigma_2^{0} \)-absorbing.
7. Some problems in dimension theory

In this section we address some problems related to distinguishing between certain classes of infinite-dimensional compacta intermediate between the class cd of countable-dimensional compacta and the class wid of weakly-infinite-dimensional compacta:

\[ \text{fd} \subset \text{cd} \subset \sigma \text{hd} \subset \text{trt} \subset \text{C} \subset \text{wid} \]

In this diagram, by fd and C we denote the classes of finite-dimensional compacta and compact with the property C. The classes σhd and trt are less known and consist of σ-hereditarily disconnected and trt-dimensional compacta, respectively. A topological space X is called σ-hereditarily disconnected if X can be written as the countable union of hereditarily disconnected subspaces.

The definition of trt-dimensional compacta is a bit longer and relies on the transfinite dimension trt introduced by Arenas, Chatyrko, and Puertas in [4]. For a space X they put

1. \( \text{trt}(X) = -1 \) iff \( X = \emptyset \);
2. \( \text{trt}(X) \leq \alpha \) for an ordinal \( \alpha \) iff each closed subset \( A \subset X \) with \( |A| \geq 2 \) can be separated by a closed subset \( B \subset A \) with \( \text{trt}(B) < \alpha \).
3. \( \text{trt}(X) = \alpha \) if \( \text{trt}(A) \leq \alpha \) and \( \text{trt}(A) \not\leq \beta \) for any \( \beta < \alpha \).

A space X is called \text{trt}-dimensional if \( \text{trt}(X) = \alpha \) for some ordinal \( \alpha \).

In [4] it was proved that each \text{trt}-dimensional compactum is a C-space, which gives the inclusion \text{trt} \subset \text{C}. The inclusion \( \sigma \text{hd} \subset \text{trt} \) was proved in [31] with help of a game characterization of \text{trt}-dimensional spaces.

The classes cd and \( \sigma \text{hd} \) of countable-dimensional and \( \sigma \)-hereditarily disconnected compacta are distinguished by the famous Pol’s compactum. We do not know if the other considered classes also are different.

**Question 7.1.** Is each \text{trt}-dimensional compactum \( \sigma \)-hereditarily disconnected? Is each \( \text{C} \)-compactum \text{trt}-dimensional?

Recently, P. Borst [35] announced an example of a weakly infinite-dimensional compact metric space which fails to be a \( \text{C} \)-space, thus distinguishing the classes wid and C.

Some immediate questions still are open for the transfinite dimension trt.

**Question 7.2.** Is the ordinal \( \text{trt}(X) \) countable for each \text{trt}-dimensional metrizable compactum \( X \)?

8. Homological methods in dimension theory

In this section we discuss some problems lying in the intersection of Infinite-Dimensional Topology, Dimension Theory, and Algebraic Topology. With help of (co)homologies we shall define two new dimension classes \( AZ_\infty \) and hsp of compacta including all trt-dimensional compacta.

The starting point is the homological characterization of \( Z_n \)-sets in ANRs due to Daverman and Walsh [49]: a closed subset \( A \) of an ANR-space \( X \) is a \( Z_n \)-set...
in $X$ for $n \geq 2$ if and only if $A$ is a $Z_2$-set in $X$ and $H_k(U, U \setminus A) = 0$ for all $k \leq n$ and all open subsets $U \subset X$.

Having this characterization in mind we define a closed subset $A \subset X$ to be a $G$-homological $Z_n$-set in $X$ for a coefficient group $G$ if the singular relative homology groups $H_k(U, U \setminus A; G)$ are trivial for all $k \leq n$ and all open subsets $U \subset X$. If $G = \mathbb{Z}$, we shall omit the notation of the coefficient group and will speak about homological $Z_n$-sets. Thus a subset $A$ of an ANR-space $X$ is a $Z_n$-set for $n \geq 2$ if and only if it is a $Z_2$-set and a homological $Z_n$-set in $X$. Another characterization of $Z_n$-sets from [20] asserts that a closed subset $A$ of an ANR-space is a homological $Z_n$-set in $X$ if and only if $A \times \{0\}$ is a $Z_{n+1}$-set in $X \times [-1, 1]$.

It is more convenient to work with homological $Z_n$-sets than with usual $Z_n$-sets because of the absence of many wild counterexamples like wild Cantor sets in $Q$ (these are topological copies of the Cantor set in $Q$ that fail to be $Z_2$-sets, see [75]). According to an old result of Kroonenberg [64] any finite-dimensional closed subset $A \subset Q$ is a homological $Z_\infty$-set in $Q$. A more general result was proved in [20]: each closed $\text{trt}$-dimensional subset $A \subset Q$ is a homological $Z_\infty$-set.

We do not know if the same is true for other classes of infinite-dimensional spaces like $C$ or wid.

**Question 8.1.** Is a closed subset $A \subset Q$ a homological $Z_\infty$-set in $Q$ if $A$ is weakly infinite-dimensional? A is a $C$-space?

This question is equivalent to the following one.

**Question 8.2.** Let $W \subset Q$ be a closed weakly-infinite dimensional subset (with the property $C$). Is the complement $Q \setminus W$ homologically trivial?

The preceding discussion suggests introducing new dimension classes $AZ_n$ consisting of so-called absolute $Z_n$-compacta. Namely, we define a compact space $K$ to be an absolute $Z_n$-compactum if for every embedding $\varepsilon: K \to Q$ of $K$ into the Hilbert cube $Q$ the image $\varepsilon(K)$ is a homological $Z_n$-set in $Q$. Among the classes $AZ_n$ the most interesting are the extremal classes $AZ_0$ and $AZ_\infty$. Both of them are hereditary with respect to taking closed subspaces.

In fact, the class $AZ_0$ coincides with the class of all compact spaces containing no copy of the Hilbert cube and thus $AZ_0$ is the largest possible nontrivial hereditary class of compact spaces. The class $AZ_0$ is strictly larger than the class $AZ_1$: the difference $AZ_0 \setminus AZ_1$ contains all hederitarily indecomposable continua $K \subset Q$ separating the Hilbert cube $Q$ (such continua exist according to [33]). Observe also that $AZ_\infty = \bigcap_{n \in \omega} AZ_n$.

**Question 8.3.** What can be said about the classes $AZ_n$ for $n \in \mathbb{N}$. Are they hereditary with respect to taking closed subspaces? Are they pairwise distinct?

The class $AZ_\infty$ is quite rich and contains all $\text{trt}$-dimensional compacta. Besides being absolute $Z_\infty$-compacta, $\text{trt}$-dimensional compacta have another interesting property: they contain many (co)homologically stable points. A point $x$ of a space $X$ will be called homologically (resp. cohomologically) stable if for some $k \geq 0$ the singular homology group $H_k(X, X \setminus \{x\})$ (resp. Čech cohomology group
Closed nonempty subspaces have cohomologically stable points.

\[ H^k(X, X \setminus \{x\}) \] is not trivial. For locally contractible spaces both notions are equivalent due to the duality between singular homologies and Čech cohomologies in such spaces. But it seems that Čech cohomologies work better beyond the class of locally contractible spaces.

According to [31], each \( \text{trt} \)-dimensional space contains a (co)homologically stable point and by [19] or [20] the same is true for every locally contractible \( C \)-compactum. Local contractibility is essential for the proof of the latter result and we do not known if it can be removed.

**Question 8.4.** Has each weakly infinite-dimensional \((C-)\)compactum a cohomologically stable point?

According to a classical result of Aleksandrov, each compact space \( X \) of finite cohomological dimension \( \dim_Z(X) \) contains a cohomologically stable point. This implies that the class \( \text{fd}_Z \) of compacta with finite cohomological dimension lies in the class \( \text{hsp} \) of compacta whose any closed subspace has a cohomologically stable point. The class \( \text{fd}_Z \) is also contained in the class \( \text{afd} \) of all almost finite-dimensional compacta, where a space \( X \) is called *almost finite-dimensional* if there is \( n \in \omega \) such that each closed finite-dimensional subspace \( F \subset X \) has dimension \( \dim(F) \leq n \). By [14], each almost finite-dimensional compactum is an absolute \( Z_\infty \)-space. Figure 2 describes the (inclusion) relations between the considered classes of compacta (the arrow \( x \rightarrow y \) means that \( x \subset y \)).

It follows from [2] (see also [56]) that the classes \( \text{fd}_Z \) and \( C \) are *orthogonal* in the sense that \( \text{fd}_Z \cap C = \text{fd} \). Is the same true for the intersection \( \text{fd}_Z \cap \text{wid} \)?

**Question 8.5** (Dranishnikov). *Is a weakly infinite-dimensional compact space finite-dimensional if it has finite cohomological dimension?*

A similar question concerns the class \( \text{afd} \) of almost finite-dimensional compacta. It is known [14] (and can be easily shown by transfinite induction) that \( \text{afd} \cap \text{trt} = \text{fd} \). Is the same true for the intersection \( \text{afd} \cap C \)? More precisely:

**Question 8.6.** *Is a compact metrizable \( C \)-space finite-dimensional if it is almost finite dimensional?*

Another interesting class from the diagram is the class \( \text{hsp} \) of compacta all whose closed nonempty subspaces have cohomologically stable points.
Question 8.7. What is the relation between the class $hsp$ and other dimension classes from the diagram? In particular, has a (locally contractible) compact space $X$ a cohomologically stable point if $X$ is almost finite-dimensional? weakly infinite-dimensional? an absolute $Z_{\infty}$-space?

Question 8.8. Is a compact space $X$ an absolute $Z_{\infty}$-space if

- all closed subspaces of $X$ have a cohomologically stable point?
- all almost finite-dimensional closed subspaces of $X$ are finite-dimensional?

We have defined absolute $Z_{\infty}$-compacta with help of their embedding into the Hilbert cube. What about embeddings into other spaces resembling the Hilbert cube?

Question 8.9. Let $A$ be a compact subset of an absolute retract $X$ whose all points are homological $Z_{\infty}$-points. Is $A$ a homological $Z_{\infty}$-set in $X$ if $A$ is an absolute $Z_{\infty}$-space?

Compact absolute retracts whose all points are homological $Z_{\infty}$-points seem to be very close to being Hilbert cubes. By [20] all such spaces fail to be $C$-spaces and have infinite cohomological dimension with respect to any coefficient group.

Question 8.10. Let $X$ be a compact absolute retract whose all points are homological $Z_{\infty}$-points. Is $X$ strongly infinite-dimensional? Is $X \times [0,1]^2$ homeomorphic to the Hilbert cube? Is $X$ homeomorphic to $Q$ if $X$ has DDP, the Disjoint Disks Property?

In light of this question we should mention an example of a fake Hilbert cube constructed by Singh [70]. He constructed a compact absolute retract $X$ such that (i) all points of $X$ are homological $Z_{\infty}$-points, (ii) $X \times [0,1]^2$ and $X \times X$ are homeomorphic to $Q$ but (iii) $X$ contains no proper closed ANR-subspace of dimension greater than one.

A bit weaker question of the same spirit asks if the Square Root Theorem holds for the Hilbert cube.

Question 8.11. Is a space $X$ homeomorphic to the Hilbert cube if $X$ has DDP and $X^2$ is homeomorphic to $Q$.

Let us note that for the Cantor and Tychonov cubes the Square Root Theorem is true, see [26].

The Singh’s example shows that the class $AZ_0$ of absolute $Z_0$-compacta is not multiplicative. An analogous question for the class $AZ_{\infty}$ is open.

Question 8.12. Is the class $AZ_{\infty}$ closed with respect to taking finite products?

It should be noted that the product $X \times Y$ of a compact absolute $Z_{\infty}$-space $X$ and a trt-dimensional compact space $Y$ is an absolute $Z_{\infty}$-space, see [14].

9. Infinite-dimensional spaces in nature

The Gromov–Hausdorff distance between compact metric spaces $(X_1,d_1)$ and $(X_2,d_2)$ is the infimum of the Hausdorff distance between the images of isometric
embeddings of these spaces into a metric space. Let $GH$ denote the set of all compact metric spaces (up to isometry) endowed with the Gromov–Hausdorff metric. We call $GH$ the Gromov–Hausdorff hyperspace. It is well-known that $GH$ is a complete separable space.

**Question 9.1.** Is the Gromov–Hausdorff hyperspace homeomorphic to $\ell^2$?

**Question 9.2.** Is the subspace of the Gromov–Hausdorff hyperspace consisting of all finite metric spaces homeomorphic to $\sigma$?

**Question 9.3.** What is the Borel type of the subspace of the Gromov–Hausdorff hyperspace consisting of all compact metric spaces of dimension $\leq n$?

A convex metric compactum is a convex compact subspace of a normed space.

**Question 9.4.** Is the subspace of the Gromov–Hausdorff hyperspace consisting of all convex metric compacta homeomorphic to $\ell^2$?

**Question 9.5.** Is the subspace of the Gromov–Hausdorff hyperspace consisting of all convex finite polyhedra homeomorphic to $\sigma$?

A tree is a connected acyclic graph endowed with the path metric.

**Question 9.6.** Is the subspace of the Gromov–Hausdorff hyperspace consisting of all finite trees homeomorphic to $\sigma$?

For any metric space $X$, one can consider the Gromov–Hausdorff space $GH(X)$, the subspace of $GH$ consisting of the (isometric copies of the) nonempty compact subsets of $X$. Note that the properties of $GH(X)$ can considerably differ from those of the Hausdorff hyperspace exp $X$: as L. Bazylevych remarked, the space $GH(X)$ need not be zero-dimensional for zero-dimensional $X$.

**Question 9.7.** Is the Gromov–Hausdorff hyperspace $GH([0,1])$ homeomorphic to the Hilbert cube?

Recall that the Banach–Mazur compactum $Q(n)$ is the space of isometry classes of $n$-dimensional Banach-spaces. The space $Q(n)$ is endowed with the distance $d(E,F) = \log \inf \{\|T\| \cdot \|T^{-1}\| : T : E \to F$ is an isomorphism$.\}$. Let $\{\text{Eucl}\} \in Q(n)$ denote the Euclidean point to which corresponds the isometry class of standard $n$-dimensional Euclidean space. It is proved in [1] (see also [3]) that the space $Q_E(n) = Q(n) \setminus \{\text{Eucl}\}$ is a $Q$-manifold.

**Question 9.8.** Are the $Q$-manifolds $Q_E(n)$ and $Q_E(m)$ homeomorphic for $n \neq m$?

**Question 9.9.** Is the subspace of the Banach–Mazur compactum $Q_{\text{pol}}(n)$ consisting of classes of equivalence of polyhedral norms a $\sigma$-manifold? If so, is the pair $(Q_E(n), Q_{\text{pol}}(n))$ a $(Q, \sigma)$-manifold?

**Question 9.10.** Is the subspace of $Q_E(n) = Q(n) \setminus \{\text{Eucl}\}$ consisting of classes of equivalence of (smooth) strictly convex norms an $\ell^2$-manifold?
Question 9.11. Is there a topological field homeomorphic to the Hilbert space $\ell^2$?

Let $(X, d)$ be a complete metric space. By $\text{CL}_W(X)$ we denote the set of all nonempty closed subsets in $X$ endowed with the Wijsman topology $\tau_W$ generated by the weak topology $\{d(x, \cdot) : x \in X\}$.

Question 9.12. Let $X$ be a Polish space. What is the Borel type of the subspace $\{A \in \text{CL}_W(X) : \dim A \geq n\}$?

Question 9.13. Characterize metric spaces $X$ whose hyperspace $\text{CL}_W(X)$ is an ANR.

Some partial results concerning the latter question can be found in [65]. For a metric space $X$ by $\text{Bdd}_H(X)$ we denote the hyperspace of closed bounded subsets of $X$ endowed with the Hausdorff distance.

A metric space $(X, d)$ is called almost convex if for any points $x, y \in X$ with $d(x, y) < s + t$ for some positive reals $s, t$ there is a point $z \in X$ with $d(x, z) < s$ and $d(z, y) < t$. In particular, each subspace of the real line is almost convex.

By [47] or [66] for each almost convex metric space $X$ the hyperspace $\text{Bdd}_H(X)$ is an ANR. We do not know if the converse is true.

Question 9.14. Let $X$ be a metric space whose hyperspace $\text{Bdd}_H(X)$ is an ANR. Is the topology (the uniformity) of $X$ generated by an almost convex metric?

Metric spaces $X$ whose hyperspaces $\text{Bdd}_H(X)$ are ANRs were characterized in [29]. This characterization implies that the hyperspace $\text{Bdd}_H(X_{\#})$ of the one-dimensional subspace

$$X_{\#} = \{(x_n) \in c_0 : (\exists n \in \omega)(\forall i \neq n) x_i \in \frac{1}{t} \mathbb{Z}\}$$

of the Banach space $c_0$ is an ANR.

Question 9.15. Is the topology (the uniformity) of the space $X_{\#}$ generated by an almost convex metric?

A metric space $X$ is defined to be an absolute neighborhood uniform retract (briefly ANUR) if for any metric space $Y \supset X$ there is a uniformly continuous retraction $r : O_\varepsilon(X) \to X$ defined on an $\varepsilon$-neighborhood of $X$ in $Y$. It is known that each uniformly convex Banach space $X$ is an ANUR. In particular, the Hilbert space $\ell^2$ is an ANUR.

Question 9.16. Is $\text{Bdd}_H(\ell^2)$ an absolute neighborhood uniform retract?

It is known that $\text{Bdd}_H(\ell^2)$ is an ANR [47] and the closed subspace of $\text{Bdd}_H(\ell^2)$ consisting of closed bounded convex subsets of $\ell^2$ is an ANUR, see [32].

References

§56. Banakh, Cauty, Zarichnyi, Open problems in infinite-dimensional topology


Classical dimension theory

Vitalij A. Chatyrko

Introduction

Dimension theory has a long history which is very well described in different books and surveys (some of those published after 1990 are \[1, 2, 19-21, 28, 40, 50, 53, 55\]). This appeared about 100 years ago as a theory of an integer-valued topological invariant whose values on each simple geometric figure coincided with the number assigned in geometry to this figure as a dimension. The invariant was called dimension and at first was considered on compact metrizable spaces, in brief CMS. At the beginning there were three approaches to define the dimension notated by \(\dim\), \(\text{Ind}\) and \(\text{ind}\). They were based on very natural geometric observations and could be later easily extended outside the class of CMS. It was observed almost from the beginning that \(\dim\), \(\text{Ind}\) and \(\text{ind}\) could disagree for general spaces. Mathematicians started to talk about dimension functions or merely dimensions. But the main challenge for the dimension theory at this time was to study the dimension in the class of CMS. It was difficult to evaluate the dimension there. So topologists continued to look for more effective approaches which could involve the use of algebra. This led in particular to cohomological dimensions \(\dim_G\). But would \(\dim_G\) be the same dimension as defined by \(\dim\), \(\text{Ind}\) and \(\text{ind}\) for CMS? The answer was “no” and came from A. Dranishnikov \[12\]. After that one realized that there was not just one dimension in the class of CMS. In fact, there were many dimension functions which could be objects of study, and to restrict the dimension theory only to the class of CMS no longer made sense. Nowadays the dimension theory is huge and consists of different parts which have influence on each other. One can name as examples “classical dimension theory” (the references above), “algebraic dimension theory” (\[13, 16, 17, 57\]), “asymptotic dimension theory” (\[14\]). This division is motivated by subjects of studies, different methods and applications outside the dimension theory. This survey is devoted to the classical dimension theory, strictly speaking, to some of its parts. We will recall results obtained after 1990 and consider problems (sometimes well known) which could probably be solved without the use of deep methods of algebra or geometry (one can find many other interesting problems in the references mentioned above).

Our terminology mostly follows \[18, 19\] and most of our uncited remarks can be found in \[19\]. All spaces considered are assumed to be regular \(T_1\). We define here the covering dimension of a completely regular space \(X\), \(\dim X\), as follows. \(\dim X \leq n\) if each finite cover of \(X\) by functionally open sets has a finite refinement by functionally open sets such that each point belongs to at most \(n + 1\) of them. The large inductive dimension of a space \(X\), \(\text{Ind} X\), is defined inductively by the following way. \(\text{Ind} X = -1\) if and only if \(X = \emptyset\). \(\text{Ind} X \leq n\) if every closed subset \(A\) of \(X\) has arbitrarily tight open neighborhoods \(U\) with \(\text{Ind} \text{Bd} U \leq n - 1\), where \(\text{Bd} C\) denotes the boundary of a set \(C\). (If \(X\) is normal then, equivalently, the set \(A\)
can be separated from the complement of \( U \) by a partition \( C \) with \( \text{Ind} C \leq n - 1 \).

One can get the definition of the *small inductive dimension* of a space \( X, \text{ind} X \), from the definition of \( \text{Ind} X \) by replacing the set \( A \) by a point. It is clear that \( \text{ind} X \leq \text{Ind} X \); \( \text{Ind} X = 0 \) implies the normality of \( X \) and \( \dim X = 0 \); \( \dim X = 0 \) implies \( \text{ind} X = 0 \) and, if \( X \) is normal, even \( \text{Ind} X = 0 \).

### Coincidence of dimensions

Recall that for a space \( X \) we have the *Urysohn identity*

\[
\dim X = \text{Ind} X = \text{ind} X
\]

if \( X \) is separable metrizable (in brief, SM), and \( \dim X \leq \text{Ind} X = \text{ind} X \) if \( X \) is Lindelöf and perfectly normal. A space \( X \) is called *cosmic* if \( X \) is a continuous image of a SM space. Any cosmic space \( X \) is evidently Lindelöf and perfectly normal. In [5] Charalambous, following the way developed by Delistathis and Watson [10], constructed in ZFC a cosmic space \( C \) that despite being the union of countably many subspaces of the square \( I^2 \), has \( \dim C = 1 \) and \( \text{ind} C = 2 \). (Independently, A. Dow and K.P. Hart ([11]) using the strategy from [10], presented under the assumption of Martin’s Axiom for \( \sigma \)-centered partial orders a cosmic space with \( \dim = 1 \) and \( \text{ind} \geq 2 \).)

#### Question 1. How large can the gap between \( \text{ind} \) and \( \dim \) be for cosmic spaces?

Recall (cf. [10] (resp. [20])) that by a theorem of Nagami (resp. Leibo) for a paracompact space \( X \) we have \( \dim X = \text{Ind} X \) if \( X \) is the union of countably many closed metrizable subspaces (resp. the closed image of a metrizable space). So UID is valid for any space which is either the union of countably many closed SM subspaces or the closed image of a SM space. Recall (cf. [45]) that every perfectly normal union of finitely many metrizable (resp. SM) subspaces is the union of countably many closed metrizable (resp. SM) subspaces.

#### Question 2. Does UID hold for quotient images of SM spaces?

Due to Roy we know that there exist metrizable spaces with \( \text{ind} = 0 \) and \( \text{Ind} = 1 \). Later Mrówka, Ostaszewski and Kulesza simplified his construction. Thus Kulesza in [33] presented in ZFC a complete \( N \)-compact metric space \( K \) having weight \( K = \omega_1 \) such that \( \text{Ind} K^n = 1 \) for any \( n \). Recall (cf. [42]) that by a result of Katetov–Morita if every completion \( X^* \) of a metric space \( X \) has \( \text{ind} X^* \geq k \) for some integer \( k \) then \( \text{Ind} X \geq k \). In [42] Mrówka presented in ZFC a non-complete metric space \( \nu \mu_0 \) such that \( \text{ind} \nu \mu_0 = 0 \), and showed that if we additionally assume his special set-theoretic axiom \( S(\delta_0) \) then for \( n = 1, 2 \) every completion of \( (\nu \mu_0)^n \) contains an \( n \)-dimensional cube \( I^n \), and we have \( \text{Ind}(\nu \mu_0)^n = n \) (the case \( n \geq 3 \) was proved by Kulesza [34]). On the contrary if we assume \( \text{CH} \) then \( \text{Ind}(\nu \mu_0)^n \geq 1 \) for all \( n \) ([43]).

#### Question 3. Does there exist a complete \((N\text{-}compact)\) metric space \( X \) (having weight \( X = \omega_1 \)) such that \( \text{Ind} X > 1 \) and \( \text{ind} X = 0 \)?
Observe that it is still an open question if there exists in ZFC a (complete) (N-compact) metric space X (having weight \( X = \omega_1 \)) such that \( \dim X > 1 \) and \( \ind X = 0 \). Many interesting problems on this subject one can find in [43].

Recall that for a compact space X we have \( \dim X \leq \ind X; \dim X = \Ind X \) if \( \dim X = 0 \); \( \ind X = \Ind X \) if \( \ind X = 1 \), and (cf. [20, 21]) there is a lot of examples of compact spaces with noncoinciding dimensions \( \dim, \ind \) and \( \Ind \). Recently Pasynkov in [49] presented for each \( n \geq 2 \) a Dugundji (resp. homogeneous) compact space \( D_n \) (resp. \( H_n \)) with \( \dim = 1 \) and \( \ind = n \). A compact space X is Dugundji if for any compact space Y with \( \dim Y = 0 \) every continuous mapping from any its closed subset to X has a continuous extension to the whole Y. A space X is homogeneous if for each pair of points \( x, y \) in X there is a homeomorphism \( h: X \to X \) such that \( h(x) = y \). Recall (cf. [49]) that by a theorem of Fedorchuk for any Dugundji compact space we have \( \ind = \Ind \), but for a homogeneous compact space this is unknown. A compact space X is algebraically homogeneous if there exists a topological group G and its closed subgroup H such that \( X = G/H \). By a theorem of Pasynkov (cf. [50]) if G is locally compact then UID holds in X.

**Question 4.** Does UID hold for any algebraically homogeneous compact space? 1320?

It is unknown if \( H_n \) is algebraically homogeneous.

Since late sixties of 20th century (cf. [20]) we know due to Filippov, Lifanov and Pasynkov that there are compact spaces with \( \ind \neq \Ind \). In particular, there exists a sequence of compact spaces \( \{X_i\}_{i=2}^{\infty} \) (resp. \( \{Y_i\}_{i=2}^{\infty} \)) such that for each i, \( \ind X_i = i \), \( \Ind X_i = 2i - 1 \) and \( \dim X_i = 1 \) (resp. \( \ind Y_i = \dim Y_i = i \), \( \Ind Y_i = i + 1 \)).

**Question 5.** Does there exist for each \( n \geq 5 \) a compact space \( Z_n \) such that \( \dim Z_n = 1 \), \( \ind Z_n = 2 \) and \( \Ind Z_n = n \)? 1321?

Such \( Z_4 \) was constructed by Kotkin ([31]). The positive answer to this question would show that for any integers \( n, m, p \) such that \( 1 \leq n \leq m \leq p \) there exists a compact space \( X_{n, m, p} \) with \( \dim X_{n, m, p} = n \leq \ind X_{n, m, p} = m \leq \Ind X_{n, m, p} = p \).

The definition of Dimensionsgrad, Dg, can be obtained from the definition of Ind for normal spaces via partitions by replacing the word “partition” with the word “cut”. In [25] Fedorchuk, Levin, Scepin proved that for any metrizable compact space X we have \( \Dg X = \dim X \), but surprisingly for complete SM spaces the dimensions can differ as was showed by Fedorchuk and van Mill (cf. [40]). It is known (cf. [21]) that \( \dim X \leq \Dg X \leq \ind X \) for any compact space. A compact space X is snake-like if for any open cover \( \alpha \) of X there exists an open refinement \( \beta = \{U_j\}_{j=1}^{\infty} \) such that \( U_i \cap U_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \).

**Question 6.** Does the equality \( \Dg X = 1 \) hold for any snake-like compact space X? 1322?

In [6] Charalambous presented for each \( n > 1 \) a snake-like compact space \( C_n \) such that \( 1 \leq \Dg C_n < \ind C_n = \Ind C_n = n \) but the value \( \Dg C_n \) remained unknown. There are few examples of compact spaces where \( \dim, \Dg \) and \( \ind \)
disagree. Thus Chatyrko and Fedorchuk ([23]) constructed a compact space $X$ with $\dim X = 1 < \text{Dg} X = 2 < \ind X = \Ind X = 3$. But we do not still know if there exists a compact space where $\ind < \text{Dg}$ or all $\dim$, $\text{Dg}$, $\ind$ and $\Ind$ disagree.

Recall that if a space $X$ is normal then $\dim X \leq \Ind X$. For a normal $n$-manifold $M$ we have additionally $n = \ind M \leq \dim M$. It is known (cf. [20]) that if a manifold $M$ is weakly paracompact then it is SM. The first example of a manifold with non-coinciding dimensions was presented by Fedorchuk and Filippov in [24]. Namely, they constructed in CH for each $n \geq 3$ a normal countably compact $n$-manifold $M_n$ such that $n = \dim M_n < \Ind M_n = 2n - 2$. Later Fedorchuk (cf. [20]) constructed also in CH for any $m, n$ such that $4 \leq n < m$, a perfectly normal separable $n$-manifold $M^{n,m}$ with $m - 1 \leq \dim M^{n,m} \leq m < m + n - 3 \leq \Ind M^{n,m} \leq m + n - 1$.

**Question 7.** Do there exist in ZFC (separable) (countably compact) $n$-manifolds $M$ where UID does not hold (dim $M \neq n$)?

It is unknown if there exist a 2-manifold $M$ with $\Ind M > 2$, a 3-manifold $N$ with $\dim N > 3$ and an $n$-manifold $M_n$ such that $\dim M_n = \Ind M_n > n$ for each $n$.

**Addition theorems for dimensions** $\dim$, $\Ind$, $\ind$

Recall that if a normal space $X$ is the union of two closed subsets $A, B$ then $\Ind X \leq \ind A + \Ind B$, and there exists a compact space $L = \bigcup_{i=1}^2 L_i$ such that $\Ind L = 2$ and for each $i$ the set $L_i$ is closed in $L$ and $\Ind L_i = 1$. In [31] Kotkin constructed a compact space $K = \bigcup_{i=1}^3 K_i$ such that $\ind K = 3$ and for each $i$ the set $K_i$ is closed in $K$ and $\Ind K_i = 1$.

**Question 8.** Does there exist for each $n = 4, 5, \ldots$ a compact space $X_n = \bigcup_{i=1}^n Y_{i,n}$ such that $\Ind X_n = n$ and for each $i$ the set $Y_{i,n}$ is closed in $X_n$ and $\Ind Y_{i,n} = 1$?

In [7] Chatyrko proved that if a space $X$ is the union of two closed subsets $A, B$ then $\ind X \leq \max\{\ind A, \ind B\} + 1$. Recall that for any normal space $X$ being the union of countably many closed subsets $X_i$, we have

\[(*) \quad \dim X = \max\{\dim X_i\}.\]

So if $n \geq 4$, we would have $1 = \dim X_n < 2 \leq \ind X_n \leq p + 1 < \Ind X_n = n$ for the smallest $p$ such that $n \leq 2^p$.

Let $d$ will be either $\ind$ or $\Ind$. We will say that the finite sum theorem for $d$ holds in a space $X$ (in dimension $k \geq 0$), in brief, $\text{FST}(d)$ (respectively, $\text{FST}(d, k)$), if $d(A \cup B) = \max\{d A, d B\}$ for any closed in $X$ sets $A$ and $B$ (such that $d A, d B \leq k$). For any space $X$ define $\text{FST}(d, X) = \infty$, if $\text{FST}(d)$ holds in $X$; $\min\{k \geq 0 \text{ such that } \text{FST}(d, k) \text{ does not hold in } X\}$ otherwise.

**Question 9.** Does there exist for each integer $m > 1$ a space $X$ such that $\text{FST}(d, X) = m$?
Let now \( d \) be either \( \dim, \ind \) or \( \Ind \), and \( X = A \cup B \). Recall that if \( X \) is hereditarily normal then we have the Urysohn inequality,

\[(\text{UIN}) \quad dX \leq dA + dB + 1.\]

UIN could be useful when we want to evaluate \( d \): if for a space \( Z = \bigcup_{i=0}^{n} Z_i \), where for each \( i \), \( dZ_i < 0 \), we have \( dZ = n \geq 1 \), and for any subsets \( U, V \) of \( Z \), \( d(U \cup V) \leq dU + dV + 1 \), then for any \( k \) such that \( 1 \leq k \leq n \), \( d(\bigcup_{i=0}^{k} Z_i) = k \). In [41] Mrówka showed that for any function \( f: \{1, 2, 3\} \rightarrow \{0, 1, 2, \ldots, \infty\} \) there exists a completely regular space \( X \) such that \( \dim A = f(1) \), \( \dim B = f(2) \), \( \dim X = f(3) \), and the sets \( A, B \) are countable intersections of clopen sets (moreover, \( X \) is of type \( N \cup R \), that is \( X \) is also the union of two discrete subspaces \( X_1, X_2 \), one of which is open, dense and countable, \( X \) is first countable, separable, locally compact, pseudocompact and all compact subspaces of \( X \) have \( d = 0 \)). This statement witnesses some known facts by E. Pol, R. Pol and Terasawa about \( \dim \) in completely regular spaces, concerning the failures of \((\ast)\), UIN and the monotonicity with respect to closed subsets (the last one was also demonstrated by the earlier mentioned example of Kulesza). However, if \( F \subset Z \subset Y \), where \( Y \) is normal and \( F \) is closed in \( Y \), then \( \dim F \leq \dim Z \). This implies by a standard method with help of \((\ast)\) that UIN holds for \( \dim \) for any normal space \( X \). In [59] Zambahidze proved that UIN is valid for \( \Ind \) for any normal space \( X \), where \( \text{FST(Ind)} \) holds. Let \( \Ind A = n \), \( \Ind B = m \) and \( n, m \geq 0 \). One can prove that \( \Ind X \leq mn + 2(m + n + 1) \) if \( X \) is normal; \( \ind X \leq 2 \cdot (n + m + 1) \); and \( \ind X \leq n + m + 1 \) if \( \text{FST(ind)} \) holds in \( X \).

**Question 10.** Does there exist a (normal) space \( X = A \cup B \) such that \( \ind X = 2 \) (resp., \( \Ind X = 2 \)) and \( \Ind A = \Ind B = 0 \)?

In [58] Tsereteli constructed completely regular space \( T = T_1 \cup T_2 \) such that \( \Ind T \geq 2 \), \( T_1 \) is discrete, \( T_2 \) is dense and \( \Ind T_i = 0 \) for each \( i \). Note that if \( X = \bigcup_{i=0}^{n} X_i \), where for each \( i \) the subspace \( X_i \) is either discrete or dense and has \( \ind X = 0 \) then \( \ind X \leq n \). We know due to Katětov (cf. [21], see also Oka [45] about different generalizations) that for a metrizable space \( X \) we have \( \dim X \leq n \) if and only if \( X = \bigcup_{i=0}^{n} X_i \) where \( \dim X_i = 0 \) for each \( i \).

**Question 11.** Does there exist a metrizable space \( X \) such that \( 0 < \ind X < \infty \) which is not the union of \( \ind X + 1 \) many subspaces having \( \ind X = 0 \)?

Observe (cf. [21]) that for each \( n \geq 1 \) (resp. \( \infty \)) there exists a first countable separable snake-like compact space \( S_n \) (where \( S_1 = [0, 1] \)) such that \( \ind S_n = n \) and each open subset of \( S_{n+1} \) contains a copy of \( S_n \) (resp. each closed non-trivial connected subset of \( S_\infty \) has \( \ind = \infty \)). So we can prove that for each \( n \geq 2 \) (resp. \( \infty \)), \( S_n \) is not the union of \( n \) (resp. countably many) subspaces having \( \ind = 0 \). In \( \text{CH} \) by a result of Odincov one can assume that all \( S_n \) are perfectly normal (cf. [21]).

Let \( d \) be either \( \dim \) or \( \Ind \).
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**Question 12.** Does there exist (for \( \text{dim} \) in ZFC) a hereditarily normal space \( X \) such that \( dX < \infty \) which is not the union of countably many (even \( dX + 1 \)) many subspaces having \( \text{ind} = 0 \) (even \( d = 0 \))? 

In [59] Zambakhidze presented a homogeneous paracompact space \( Z \) with \( \text{Ind} Z = 2 \) which is not the union of three subspaces of \( \text{Ind} = 0 \).

**Product theorems for dimensions** \( \text{dim}, \text{ind}, \text{Ind} \)

One can show that for any regular spaces \( X, Y \) and any \( k \geq 0 \) such that \( \text{ind} X = n, \text{ind} Y = m \) and \( \text{FST} (\text{ind}, X), \text{FST} (\text{ind}, Y) \geq k \) we have \( \text{ind} (X \times Y) \leq n + m \), if either \( n \geq 0 \), or \( m = 0 \), or \( n, m \leq k \); 2\((n + m) - k - 1\), otherwise. This is a combination of known facts by Pasynkov, Basmanov \((k = \infty)\) and Chatyrko, Kozlov \((k = 0, 1)\). Observe that for any compact space \( Z \) we have \( \text{FST} (\text{ind}, Z) \geq 1 \), and there exist due to Filippov two compact spaces \( X \) and \( Y \) such that \( \text{Ind} X = 2, \text{Ind} Y = 1 \) and \( \text{ind} (X \times Y) = 4 \).

**Question 13.** Does there exist two (compact) spaces \( X, Y \) such that \( \text{ind} X = \text{ind} Y = 1 \) (resp., 2) and \( \text{ind} (X \times Y) = 3 \) (resp., 6)?

Recall that for any completely regular space \( X \) we have \( \text{ind} (X \times I) \leq \text{ind} X + 1 \).

In [39] D. Malykhin constructed a regular space \( M \) with \( \text{ind} M = 2 \) such that \( \text{ind} (M \times I) = 4 \).

Let \( \Pi = X \times Y \) be the product of two completely regular spaces \( X, Y \). \( \Pi \) is **piecewise rectangular** if for any finite functionally open cover of \( \Pi \) there exists a \( \sigma \)-locally finite refinement consisting of clopen subsets of functionally open rectangles (= the products of functionally open subsets of \( X \) and \( Y \)). Let \( d \) be either \( \text{dim} \) or \( \text{Ind} \) and \( dX = n, dY = m \). It is known (cf. [50]) that \( d \Pi \leq n + m \) if \( \Pi \) is piecewise rectangular and either \( d = \text{dim} \) or \( d = \text{Ind} \), \( \Pi \) is normal and \( \text{FST} (\text{Ind}) \) holds in \( X \) and \( Y \). Recall (cf. [50]) that \( \Pi \) is piecewise rectangular if for example \( \Pi \) is normal and \( X \) is metrizable or the projection of \( \Pi \) onto \( X \) is closed or \( X \) is locally compact paracompact or \( \Pi \) is completely paracompact etc.

**Question 14.** Does the inequality \( \text{dim} \Pi \leq \text{dim} X + \text{dim} Y \) hold for a paracompact product \( \Pi \)?

In [48] Pasynkov showed that \( \text{Ind} \Pi < \infty \) if \( \Pi \) is normal and \( X \) is either locally compact paracompact or metrizable.

**Question 15.** Is \( \text{Ind} \Pi < \infty \) if \( \Pi \) is normal and piecewise rectangular?

Recall (cf. [50]) that \( \text{dim} \Pi = 0 \) if and only if \( \Pi \) is piecewise rectangular and \( \text{dim} X = \text{dim} Y = 0 \), and there are due to Wage, Przymusiński, Tsuda, E. Pol, Engelking, Kozlov different examples of normal products \( \Pi \) with zero-dimensional in the sense of \( \text{dim} \) factors such that \( \text{dim} \Pi > 0 \). Thus Kozlov in [32] applying Przymusiński’s technique showed that for any positive integers \( k, m, n \) such that \( k < m \) there exists a first countable space \( K \) satisfying: (i) \( K^s \) is Lindelöf if and only if \( s < k \); (ii) \( K^s \) is collectionwise normal and countably paracompact if \( s \leq m \); (iii) \( \text{dim} K^s = 0 \) for \( s < m \); (iv) \( \text{dim} K^m = \text{Ind} K^m = n \); moreover for \( k = 1 \) we can assume that \( K \) is locally compact and locally countable.
Question 16. Does there exist a normal product $\Pi$ with $\dim X = \dim Y = 0$ where $\dim$ and Ind disagree?

In [27] Hattori refined some earlier result of Kulesza: for each pair $n \leq d$ there is a non-complete subgroup $G_{n,d}$ of $R^{n+1}$ satisfying: $\dim G_{n,d} = n$ and $\dim(G_{n,d})^\omega = d$. It is unknown if there are SM complete groups with such dimensional properties.

In [15] Dranishnikov improved UIN for compact metrizable spaces $X$ such that $\dim X^2 = 2 \dim X$. Namely, if such $X$ is the union $X_1 \cup X_2$, then $\dim X \leq \dim(X_1 \times X_2) + 1$. He also showed that the weaker inequality with 1 replaced by 2 holds without the mentioned restriction.

Question 17. Does there exist a (separable) metrizable space $X = X_1 \cup X_2$ such that $\dim X > \dim(X_1 \times X_2) + 1$?

In [15] Dranishnikov presented a metrizable compact space $X = X_1 \cup X_2$ such that $\dim X > \dim(Y_1 \times Y_2) + 1$ for any compacta $Y_1 \subset X_1$ and $Y_2 \subset X_2$. Evidently, there exists a completely regular space $X$ (the earlier mentioned space $N \cup R$ of Mrówka) such that $X = X_1 \cup X_2$ and $\dim X = \infty > \dim(X_1 \times X_2) + 1 = 0 + 1 = 1$.

Compactifications

It is known that $\dim \beta X = \dim X$ if $X$ is completely regular, and $\Ind \beta X = \Ind X$ if $X$ is normal. Moreover, there exists a preserving weight compactification $bX$ (resp. $cX$) of $X$ such that $\dim bX = \dim X$ (resp. $\Ind bX \leq \Ind X$) if $X$ is completely regular (resp. normal). Recall that there exists a perfectly normal first countable space $P$ with $\ind P = 1$ each Lindelöf extension of which has $\ind = \infty$. Now it is natural to look for two classes $A$ and $B$ of spaces, where $B$ is better than $A$, and properties such that for each element from $A$ there exists its extension from $B$ preserving the properties. Thus Kimura and Morishita in [30] showed that every metrizable space has a compactification that is Eberlein compact and preserves both dim and weight (in [4] Charalambous proved that this compactification preserves also Ind). A compact space $E$ is said to be Eberlein compact if $E$ is homeomorphic to a subset of a Banach space with its weak topology. A space $U$ is universal for a class of spaces if each element of this class can be embedded in $U$. Recall (cf. [50]) that for a class $M$ (resp. $D$ or $I$) of all metrizable (resp. completely regular or normal) spaces with weight $\leq \tau$ and $\dim \leq n$ (resp. $\dim \leq n$ or $\Ind \leq n$), where $n \geq 0$, there exists an element from this class that is universal for $M$ (resp. $D$ or $I$, and the element is compact). So the Kimura–Morishita result (and the result of Charalambous as well) implies the existence of an Eberlein compact space $E_{n,\tau}$ which is universal for all metrizable spaces with $\dim \leq n$ and weight $\leq \tau$ and which has the same weight and dimensions dim and Ind.

Infinite-dimensional theory

All spaces considered here are SM. The inductive dimensions ind, Ind have natural transfinite extensions $\trind$, $\trInd$ for which $\trind \leq \trInd$. An infinite-dimensional space is countable dimensional, shortly c.d., if $X$ is the union of
countably many finite-dimensional subspaces. It is known that every space having \text{trind} < \omega_1 is c.d. and any c.d. compact space has \text{trind} < \omega_1. Recall (\text{[38]})) that there exist two functions \( \phi, \psi : \{ \alpha < \omega_1 \} \to \{ \alpha < \omega_1 \} \) such that \text{trInd} \( X \leq \phi(\text{trind} X) \) and \( \psi(\text{trInd} X) \leq \text{trind} X \) for any compact space \( X \). Moreover, for each \( \alpha < \omega_1 \) there exist compact spaces \( X_\alpha \) and \( Y_\alpha \) such that \text{trind} \( X_\alpha = \text{trInd} Y_\alpha = \alpha \), \text{trInd} \( X_\alpha = \phi(\alpha) \) and \text{trind} \( Y_\alpha = \psi(\alpha) \). Smirnov’s compacta \( S^\alpha, \alpha < \omega_1 \), are defined as follows. For each integer \( n \geq 0 \) the space \( S^n \) is the Euclidean \( n \)-cube \( I^n \), \( S^{\alpha+1}_n = S^\alpha \times I \) and, for any limit \( \alpha \), \( S^\alpha \) is the one-point compactification of the free union of \( S^\beta \) with \( \beta < \alpha \). It is known that for each \( \alpha \), \text{trInd} \( S^\alpha = \alpha \). In [7] Chatyrko improved an earlier result of Luxemburg to the following effect: for each \( m \geq 0 \) and any limit \( \lambda < \omega_1 \), \( \text{trind} S^{\lambda+2^{m-1}} \leq \lambda + m \). Recall (\text{[38]})) that for each \( \lambda \) such that \( \phi(\lambda) = \lambda \) we have \( \text{trind} S^{\lambda+k} = \lambda + k \) for \( k = 0, 1, 2 \).

13367 **Question 18.** What is \( \text{trind} S^\alpha \) for each \( \alpha \)?

One can decompose \( S^{\lambda+n} = Y_0 \cup \cdots \cup Y_n \), where \( n \geq 1 \), into closed subsets \( Y_i \) such that for each \( i \), \( \text{trInd} Y_i = \lambda \). Using the following sum theorem: if \( X = X_1 \cup X_2 \), where \( X_1 \), \( X_2 \) are closed in \( X \), then \( \text{trind} X \leq \max \{ \text{trind} X_i \} + 1 \), \( \text{trInd} X \leq \max \{ \lambda_i \} + n_1 + n_2 + 1 \), where \( \text{trInd} X_i = \lambda_i + n_i \) for each \( i \), and the unions \( Y_0 \cup \cdots \cup Y_k \), where \( k \geq 3 \), we get a variety of compact spaces with \( \text{trind} \neq \text{trInd} \).

13367 **Question 19.** Does there exist for each \( \alpha < \omega_1 \) a compact space \( X_\alpha \) such that \( \text{trind} X_\alpha = \text{trInd} X_\alpha = \alpha \)?

Observe that if FST(\text{trind}) or FST(\text{trInd}) holds in a c.d. compact space \( X \) then \( \text{trind} X = \text{trInd} X \).

Recall that by Luxemburg’s results each space \( X \) having \( \text{trInd} X < \omega_1 \) has a compactification preserving \( \text{trInd} \), and there exists a complete space with \( \text{trind} = \omega_0 \) having no compactification preserving \( \text{trind} \).

13367 **Question 20.** Evaluate for each space \( X \), \( \min \{ \text{trind} Y : Y \text{ is a compactification of } X \} \).

We know due to R. Pol that for each \( \alpha < \omega_1 \), there exists a universal space in the class of spaces with \( \text{trind} \leq \alpha \). In [47] Olszewski proved that for any limit \( \alpha \) there is no universal space neither in the class of spaces (resp. compacta) with \( \text{trInd} \leq \alpha \) nor in the class of compacta with \( \text{trind} \leq \alpha \). The existence of universal spaces for non-limit \( \alpha \) for the mentioned cases is an open question. Let \( d \) be a transfinite dimension. A compact space \( C \) is an \(( \alpha + 1)\)-dimensional d-Cantor manifold (resp. infinite-dimensional Cantor manifold) if \( d C = \alpha + 1 \) (resp. \( \dim C = \infty \)) and no closed subspace \( F \) of \( C \) satisfying \( d F \leq \alpha \) (resp. \( \dim F < \infty \)) separates \( C \). In [46] Olszewski presented for each \( \alpha < \omega_1 \) an \(( \alpha + 1)\)-dimensional d-Cantor manifold for \( d = \text{trind} \) or \( \text{trInd} \) (in [56] Renska constructed simpler examples of \( \text{trInd} \)-manifolds which are disjoint unions of countably many closed cells and irrationals). A continuum \( X \) is hereditarily indecomposable, shortly h.i., if for any subcontinua \( A, B \) in \( X \) with nonempty intersection, either \( A \subset B \), or...
We know due to Bing (cf. [40]) that for each \( n = 1, 2, \ldots, \infty \) there exist h.i. continua with \( \dim = n \), and in each such continuum \( X \) with \( \dim X = n \) the set \( B_n(X) = X \setminus \{ \text{the union of all non-trivial subcontinua with } \dim < n \} \) is not empty if \( n < \infty \). In [54] R. Pol and Renska showed that if \( X \) is a h.i. continuum with \( 2 \leq \dim X = n < \infty \) and \( B_r(X) \) is the set of all points of \( X \) belonging to some continuum with \( \dim = r \) but avoid any non-trivial continuum with \( \dim < r \), where \( 1 \leq r \leq n \), then \( \dim B_r(X) = n - (r - 1) \), moreover \( B_n(X) \) is not of type \( G_{\delta \sigma} \) (always \( G_{\delta \sigma \delta} \)-set). A space \( E \) is weakly infinite-dimensional, shortly w.i.d., if for each sequence of pairs \( (A_i, B_i), \) \( i = 1, 2, \ldots \) of disjoint closed sets in \( E \) there are partitions \( \{L_i \} \) in \( E \) between \( A_i \) and \( B_i \) such that \( \bigcap_{i=1}^{\infty} L_i = \emptyset \), otherwise \( E \) is strongly infinite dimensional, shortly s.i.d. In [52] E. Pol and Renska constructed for each infinite \( \alpha < \omega_1 \) h.i. continua with trind or trInd equal to \( \alpha \) and demonstrated the diversity among types of the sets \( B_{\infty}(X) \) for infinite-dimensional h.i. continua \( X \) (\( B_{\infty}(X) \) can be any subset of the Cantor set, the set of irrational numbers, a 1-dimensional \( G_{\delta} \)-subset of \( X \)), for s.i.d., h.i. continua \( X \), \( B_{\infty}(X) \) is always strongly infinite-dimensional that is a corollary of a theorem of Henderson (cf. [40]) or a more recent result of Levin [35].

**Question 21.** Is there for each integer \( n \geq 2 \) an infinitesimal h.i. continuum \( X \) with \( B_{\infty}(X) = n \)?

A space \( X \) is hereditarily strongly infinite-dimensional, shortly h.s.i.d., if every subspace of \( X \) is either 0-dimensional or s.i.d. We know due to Rubin (cf. [40]) that there are h.s.i.d. continua. Recall that such spaces have to contain infinite-dimensional h.s.i.d. Cantor manifolds. In [51] E. Pol constructed a family \( \{Y_s : s \in S\} \), where \( |S| = 2^{\aleph_0} \), of h.i., h.s.i.d. Cantor manifolds such that (a) no open subset of \( Y_s \) embeds in \( Y_p \) for every \( s \neq p, s, p \in S \); (b) every embedding of \( Y_s \) into \( Y_s \) is the identity, for each \( s \in S \). A space \( X \) is C-space if for every sequence \( \{\alpha_i\}_{i=1}^{\infty} \) of open covers \( X \) there exist disjoint open collections \( \beta_1, \beta_2, \ldots \) such that \( \beta_i \) refines \( \alpha_i \) for each \( i \) and \( \bigcup_{i=1}^{\infty} \beta_i = X \). Recall that every C-compact space is w.i.d. We know due to Hattori and Yamada that the product of two C-compact spaces is a C-space and the product of w.i.d. compact space and a C-compact space is w.i.d.

**Question 22.** Is the product of two w.i.d. compact spaces w.i.d.?

The dimension \( \dim \) can be extended to transfinites by different ways. Usually one uses a characterization of \( \dim \) which is possible to extend to transfinites and considers the extension as an extension of \( \dim \). In particular, we know due to Borst two extensions of \( \dim \), \( \dim_w \) and \( \dim_C \), such that for each compact space \( X \), \( \dim_w X < \omega_1 \) (resp. \( \dim_C X < \omega_1 \)) if and only if \( X \) is w.i.d. (resp. a C-space). Recently Borst [3] constructed for each \( \alpha < \omega_1 \) a compact space \( X_\alpha \) such that \( \dim_C X_\alpha = \alpha \) and \( \dim_w X_\alpha = \omega_0 \), then he embedded all \( X_\alpha \) in a compact space \( Y \) which is w.i.d. This \( Y \) can not be a C-space.
Compactness degrees

All spaces considered here are SM. The compactness deficiency of a space $X$, def $X$, is the least integer $n$ for which $X$ has a metrizable compactification $Y$ with $\dim(Y \setminus X) \leq n$. It is known due to de Groot (cf. [1]) that def $X = 0$ if and only if $X$ is rim-compact, i.e., if every point of $X$ has arbitrarily small neighborhoods with compact boundary. In spirit of definitions of ind and Ind one defined two extensions of the rim-compactness: a small inductive compactness degree of $X$, cmp $X$, (replace the empty set in the definition of ind by a compact space) and a large inductive compactness degree of $X$, Cmp $X$, (assume for $n = -1, 0$ that Cmp $X = n$ if and only if cmp $X = n$). Recall (cf. [1]) that cmp $X = 0$ if and only if def $X \leq \dim X$ for any space $X$. We knew due to R. Pol, Kimura, Hart (cf. [1]) that there were examples of SM spaces with cmp $\neq$ Cmp but these examples were rather complicated. Spaces $Z_n$, where $n \geq 1$, are geometric cubes $I^{n+1}$ without one open $n$-dimensional face. It was known almost from the beginning (cf. [1]) that Cmp $Z_n = 0$ for each $n \geq 1$ and for $n = 1, 2$, cmp $Z_n = n$. Recently Chatyrko and Hattori ([9], $n \geq 5$), Nishiura ([44], $n = 4$), Fedorchuk ([22], $n = 3$) showed that for $n \geq 3$, $2 \leq \text{cmp} Z_n \leq m$, where $m$ is any integer satisfying $n + 1 \leq 2^m$.

Questions

23. What is cmp $Z_n$ equal to for $n \geq 4$?

There are only two examples of spaces with Cmp $\neq$ def. Namely, Kimura in [29] constructed a subspace $K$ of $R^4$ such that Cmp $K = 1$ and $2 \leq \text{def} K \leq 3$. In [36] Levin and Segal found a subspace $E$ of $R^3$ such that cmp $E = \text{Cmp} E = 1$ and def $E = 2$. We do not have any example of a space where all cmp, Cmp and def disagree.

24. Does there exist for each $n$ a SM space $X_n$ such that cmp $X_n = \text{Cmp} X_n = 1$ and def $X_n = n$?

In [8] Chatyrko, following a way developed by Hart and Kimura (cf. [1]), showed the existence for each $n, m$ such that $n \leq m$ a SM space $C_{n,m}$ with cmp $C_{n,m} = n$ and Cmp $C_{n,m} = \text{def} C_{n,m} = m$. So the positive answer to this question would show that for any integers $n, m, p$ such that $1 \leq n \leq m \leq p$ there exists a SM space $X_{n,m,p}$ with cmp $X_{n,m,p} = n \leq \text{Cmp} X_{n,m,p} = m \leq \text{def} X_{n,m,p} = p$.

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Questions on weakly infinite-dimensional spaces

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Introduction

Weak infinite dimensionality was introduced by Alexandroff in 1948 [2]. The first results in this area were obtained by Sklyarenko [20] and Levshenko [13] in 1959. A great contribution to the theory of weakly infinite-dimensional spaces was made in 1981 by R. Pol [15], who constructed an example of a compact metrizable weakly infinite-dimensional space which is not countable-dimensional.

In 1974, Haver [11] introduced the $C$ property for metric spaces and proved that every locally contractible metric space which is a union of countably many compact sets with property $C$ is an ANR space. In 1978, Addis and Gresham [1] gave a topological definition of $C$-spaces.

The $C$-spaces proved to play an important role in topology. In particular, Ancel [3] showed that any cell-like map from a compact metrizable space onto a $C$-space is a hereditary shape equivalence. Consequently, every infinite-dimensional compact $C$-space has infinite cohomological dimension $c\text{-dim}_{\mathbb{Z}}$.

One of the most important problems concerning infinite-dimensional spaces was whether any weakly infinite-dimensional compact space is a $C$-space. Recently, this problem was solved in the negative by Borst [5].

The questions considered here are related to new classes of spaces, which are intermediate between the classes of weakly infinite-dimensional spaces and $C$-spaces.

1. Definitions

For a topological space $X$, by $\text{cov}(X)$ we denote the set of all open covers of $X$. A family $\mathcal{U} = \{u_\alpha : \alpha \in A\} \subset \text{cov}(X)$ is said to be essential (in $X$) if, for any disjoint open families $v_\alpha$, where $\alpha \in A$, such that $v_\alpha$ refines $u_\alpha$ for each $\alpha$, the family $\bigcup\{v_\alpha : \alpha \in A\}$ does not cover $X$.

Let $\mathcal{P}$ be a class of open covers of topological spaces; for a space $X$, we set $\mathcal{P}(X) = \mathcal{P} \cap \text{cov}(X)$. A normal space $X$ is called a $\mathcal{P}$-$C$-space ($X \in \mathcal{P}$-$C$) if every countable family $\mathcal{U} \subset \mathcal{P}(X)$ is inessential.

This approach yields the following classes of spaces:

(1) $m$-$C$-spaces, where $m$ is an integer $\geq 2$ and $\mathcal{P}$ consists of all covers $\mathcal{U}$ with $|\mathcal{U}| \leq m$;

(2) $\infty$-$C$-spaces, where $\mathcal{P}$ consists of all finite covers;

(3) $lf$-$C$-spaces, where $\mathcal{P}$ consists of all locally finite covers;

(4) $C$-spaces, where $\mathcal{P}$ consists of all covers;

This work was financially supported by the Russian Foundation for Basic Research (project no. 06-01-00761).
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and so on.

If \( P_1 \subset P_2 \), then \( P_2-C \subset P_1-C \). Consequently,

\[
C \subset \text{lf}-C \subset \infty-C \subset \cdots \subset m-C \subset \cdots \subset 2-C.
\]

The largest member of this sequence coincides with the class \( \text{wid} \) of all weakly infinite-dimensional spaces. The space \( \omega_1 \) of all countable ordinals is an \( \text{lf}-C \)-space but not a \( C \)-space.

**Question 1.** Does there exist an \( X \in \infty-C \setminus \text{lf}-C \)?

For compact spaces, the first three members of sequence (1.1) coincide.

**Question 2.** Does the equality \( (m+1)-C = m-C \) hold in the class of compact metrizable spaces for all \( m \)?

Let \( \omega-C = \bigcap\{m-C : m \in \mathbb{N}\} \).

**Question 3.** Does the equality \( C = \omega-C \) hold in the class of compact metrizable spaces?

Because of Borst’s example of a compact metrizable space \( X \in 2-C \setminus C \), the answer to one of Questions 2 and 3 must be negative.

Yet another generalization of \( C \)-spaces is as follows. Let \( \Phi = \{F_\alpha : \alpha \in A\} \) be a discrete family of closed subsets of a space \( X \). A neighborhood \( O\Phi \) of the family \( \Phi \) is a disjoint collection \( \{OF_\alpha : \alpha \in A\} \) of neighborhoods \( OF_\alpha \) of the sets \( F_\alpha \).

A set \( \varphi = \{\Phi_\beta : \beta \in B\} \) of discrete families of closed subsets of \( X \) is said to be essential (in \( X \)) if, for any neighborhoods \( O\Phi_\beta \), the family \( \bigcup\{O\Phi_\beta : \beta \in B\} \) does not cover \( X \). A collectionwise normal space \( X \) is called a weak \( C \)-space \((X \in w-C)\) if every countable family of discrete collections of closed subsets of \( X \) is inessential. Any collectionwise normal \( C \)-space is a \( w-C \)-space.

On the other hand, any finite-dimensional countably compact noncompact space is a \( w-C \)-space but not a \( C \)-space.

**Question 4.** Is it true that any paracompact \( w-C \)-space is a \( C \)-space?

The answer to this question is unknown even for compact metrizable spaces.

If a disjoint family \( \Phi \) of closed subsets of a space \( X \) consists of \( \leq m \) members, where \( m \in \mathbb{N} \), then we say that \( \Phi \) is an \( m \)-system in \( X \). An \( \infty \)-system in \( X \) is any finite disjoint family of closed subsets of \( X \). A normal space \( X \) is called a \( w-m-C \)-space \((X \in w-m-C)\), where \( m \in \mathbb{N} \) or \( m = \infty \), if any countable family of \( m \)-systems in \( X \) is inessential. Every \( m-C \)-space is a \( w-m-C \)-space. By definition, we have \( w-2-C = \text{wid} \).

**Question 5.** Is it true that any compact metrizable \( w-m-C \)-space is an \( m \)-space?

We have the following sequence of inclusions similar to (1.1):

\[
w-C \subset w-\infty-C \subset \cdots \subset w-m-C \subset \cdots \subset w-2-C.
\]

**Question 6.** Does the equality \( w-(m+1)-C = w-m-C \) hold in the class of compact metrizable spaces for all \( m \)?
Let \( w-\omega-C = \bigcap\{w-m-C : m \in \mathbb{N}\} \).

**Question 7.** Does the equality \( \omega-C = w-\omega-C \) hold in the class of compact metrizable spaces? 1348?

**Question 8.** Is it true that any closed subspace of an If-C-space is an If-C-space? 1349?

In the class of all countably paracompact collectionwise normal spaces, the answer to Question 8 is positive.

If every \( G_\delta \)-subset of a hereditarily normal space \( X \) is an \( m-C \)-space, then every subset of \( X \) is an \( m-C \)-space.

**Question 9.** Is it true that every subset of a \( w-m-C \)-space \( X \) is a \( w-m-C \)-space provided that every \( G_\delta \)-subset of \( X \) is a \( w-m-C \)-space? 1350?

Any paracompact finite-dimensional space is a \( C \)-space [1].

**Question 10.** Is it true that any weakly paracompact finite-dimensional space is a \( C \)-space? 1351?

Recall that a normal space \( X \) is said to be 0-countable-dimensional if \( X = \bigcup\{X_i : i \in \omega\} \), where \( \dim X_i \leq 0 \).

**Proposition ([8]).** Any 0-countable-dimensional collectionwise normal hereditarily normal space is a \( w-C \)-space.

**Corollary ([8]).** Any subset of a linearly ordered continuum is a \( w-C \)-space.

**Question 11.** Is it true that any collectionwise normal finite-dimensional space is a \( w-C \)-space? 1352?

**Theorem 1 ([8]).** Any strongly paracompact space \( X \) for which \( \text{ind} X \) is defined is a \( C \)-space.

Levshenko [14] proved that if \( X \) satisfies the assumptions of Theorem 1, then \( X \in \text{wid} \).

**Question 12.** Is it true that if \( X \) is a paracompact space for which \( \text{ind} X \) is defined, then \( X \in \text{wid} \)? \( X \in C \)? 1353–1354?

This question can be strengthened as follows.

**Question 13.** Is it true that if \( X \) is a metric space with \( \text{ind} X = 0 \), then \( X \in \text{wid} \)? 1355?

Question 12 can also be strengthened in a different direction.

**Question 14.** Is it true that if \( X \) is a completely paracompact space for which \( \text{ind} X \) is defined, then \( X \in \text{wid} \)? 1356?

It is known that if \( X \) is a completely paracompact metrizable space for which \( \text{ind} X \) is defined, then \( X \) is countable-dimensional and, consequently, \( X \in C \) (this was proved by Smirnov in [21]).
2. Maps, products, and subsets

If $\mathcal{P}$ is class of a spaces and $f: X \to Y$ is a map, then $f \in \mathcal{P}$ means that $f^{-1}(y) \in \mathcal{P}$ for every $y \in Y$.

**Theorem 2 ([7]).** Let $f: X \to Y$ be a closed map from a countably paracompact (or hereditarily normal) space $X$ onto a $C$-space $Y$. Then the following assertions are valid:

1. if $f \in m$-$C$, then $X \in m$-$C$;
2. if $f \in w$-$m$-$C$, then $X \in w$-$m$-$C$.

Assertion (1) was proved for $m = 2$ by Hattori and Yamada [10].

**Question 15.** Let $X$ be a countably paracompact or hereditarily normal space admitting a closed $m$-$C$-map ($w$-$m$-$C$-map) onto a $w$-$C$-space. Is it true that $X \in m$-$C$? Respectively, $X \in w$-$m$-$C$?

**Question 16.** Given compact metrizable spaces $X$ and $Y$, is it true that

1. if $X \in m$-$C$ and $Y \in m$-$C$, then $X \times Y \in m$-$C$;
2. if $X \in w$-$m$-$C$ and $Y \in w$-$m$-$C$, then $X \times Y \in w$-$m$-$C$;
3. if $X \in w$-$C$ and $Y \in w$-$C$, then $X \times Y \in w$-$C$?

**Question 17.** Let $f: X \to Y$ be a light map of compact metrizable spaces. Is it true that

1. if $Y \in m$-$C$, then $X \in m$-$C$;
2. if $Y \in w$-$m$-$C$, then $X \in w$-$m$-$C$;
3. if $Y \in w$-$C$, then $X \in w$-$C$?

A positive answer to Question 17 would imply a positive answer to Question 16 thanks to the following theorem.

**Theorem 3 ([8]).** Suppose that $\mathcal{P}$ is one of the classes $m$-$C$, $w$-$m$-$C$, $w$-$C$, and $C$. Then any compact metrizable space $X \notin \mathcal{P}$ contains a compact space $Y \notin \mathcal{P}$ such that, for any $Z \subset Y$, either $\dim Z \leq 0$ or $Z \notin \mathcal{P}$.

This theorem was proved by Rubin [19] for $\mathcal{P} = 2$-$C$, by R. Pol [18] for $\mathcal{P} = C$ and closed $Z$, and by Levin [12] for $\mathcal{P} = C$.

**Definition.** Let $\mathcal{P}$ be a topological property. A space $X$ is said to be hereditarily non-$\mathcal{P}$ if $X \notin \mathcal{P}$ and, for every closed set $Y \subset X$,

$$\text{either dim } Y \leq 0 \text{ or } Y \notin \mathcal{P}.$$  

If this alternative holds for all subsets $Y \subset X$, then we say that $X$ is a strongly hereditarily non-$\mathcal{P}$ space.

Let h-non-$\mathcal{P}$ (sh-non-$\mathcal{P}$) denote the class of all (strongly) hereditarily non-$\mathcal{P}$ spaces.

**Question 18.** Let $\mathcal{P}$ be one of the classes $m$-$C$, $w$-$m$-$C$, $w$-$C$, and $C$, and let $X$ and $Y$ be compact metrizable spaces. Is it true that
(1) \( X, Y \in \text{h-non-}\mathcal{P} \implies X \times Y \in \text{h-non-}\mathcal{P} \);
(2) \( X, Y \in \text{sh-non-}\mathcal{P} \implies X \times Y \in \text{sh-non-}\mathcal{P} \);
(3) \( X \in \text{h-non-}\mathcal{P} \implies X^2 \in \text{h-non-}\mathcal{P} \);
(4) \( X \in \text{sh-non-}\mathcal{P} \implies X^2 \in \text{sh-non-}\mathcal{P} \).

**Question 19.** Let \( \mathcal{P} \) be one of the classes \( m\text{-C}, \ w\text{-m}\text{-C}, \ w\text{-C}, \) and \( C \). Is it true that \( \text{Comp} \cap (\text{h-non-}\mathcal{P}) \subset \text{sh-non-}\mathcal{P} \)?

**Definition.** A space \( X \) is said to be strongly hereditarily (hereditarily) non-1 dim-space, \( X \in \text{sh-non-1 dim} \) (\( X \in \text{h-non-1 dim} \)), if \( \dim X \geq 2 \) and, for every (closed) set \( Y \subset X \), either \( \dim Y \leq 0 \) or \( \dim Y \geq 2 \).

**Question 20.** Let \( X \) and \( Y \) be compact metrizable spaces. Is it true that

(1) \( X, Y \in \text{h-non-1 dim} \implies X \times Y \in \text{h-non-1 dim} \);
(2) \( X, Y \in \text{sh-non-1 dim} \implies X \times Y \in \text{sh-non-1 dim} \);
(3) \( X \in \text{h-non-1 dim} \implies X^2 \in \text{h-non-1 dim} \);
(4) \( X \in \text{sh-non-1 dim} \implies X^2 \in \text{sh-non-1 dim} \).

A positive answer to Question 20(4) would give a positive answer to van Mill’s problem \cite{17}, Question 414, which can be formulated as follows.

**Question 21.** Does there exist an infinite-dimensional compact space \( X \) such that \( X^n \in \text{sh-non-1 dim} \) for all positive \( n \)?

Note that there exists no example of infinite-dimensional compact metrizable spaces \( X \) and \( Y \) such that \( X \times Y \) contains no one-dimensional compact sets. At the same time, under the continuum hypothesis, there exists an infinite compact space \( X \) such that, for any positive integer \( n \), all infinite closed subspaces of \( X^n \) are strongly infinite-dimensional \cite{9}.

**Question 22.** Is it true that any compact metrizable space containing one-dimensional subsets contains a compact one-dimensional subset?

### 3. Transfinite dimensions

#### 3.1. The ordinal number \( \text{Ord} \).

In this section, we recall Borst’s definition from \cite{4}. Let \( L \) be an arbitrary set. By \( \text{Fin} L \) we denote the collection of all finite nonempty subsets of \( L \).

Let \( M \) be a subset of \( \text{Fin} L \). For \( \sigma \in \{\emptyset\} \cup \text{Fin} L \), we set \( M^\sigma = \{\tau \in \text{Fin} L : \sigma \cup \tau \in M, \sigma \cap \tau = \emptyset\} \). For \( a \in L \), we denote the set \( M^{\{a\}} \) by \( M^a \).

**Definition.** The ordinal number \( \text{Ord} M \) is defined by induction as follows.

- \( \text{Ord} M = 0 \) if and only if \( M = \emptyset \);
- \( \text{Ord} M \leq \alpha \) if and only if \( \text{Ord} M^a < \alpha \) for every \( a \in L \);
- \( \text{Ord} M = \alpha \) if and only if \( \text{Ord} M \leq \alpha \) and it is not true that \( \text{Ord} M < \alpha \);
- \( \text{Ord} M = \infty \) if and only if \( \text{Ord} M > \alpha \) for every ordinal \( \alpha \).

For an integer \( m \geq 2 \), we set \( \text{cov}_m(X) = \{u \in \text{cov}(X) : |u| \leq m\} \) and \( \text{cov}_\infty(X) = \bigcup_m \text{cov}_m(X) \).
For an integer $m \geq 2$ and for $m = \infty$, we set $M_m(X) = \{ \sigma \in \text{Fin } \text{cov}_m(X) : \sigma \text{ is essential} \}$.

For any normal space $X$, we have

$$
(3.1) \quad \dim X \leq n \text{ if and only if } \text{Ord } M_m(X) \leq n.
$$

This is a generalization of Borst’s theorems [4, 6] for $m = 2$ and $m = \infty$.

### 3.2. Transfinite dimension $\dim_m$. For a normal space $X$, we set

$$
(3.2) \quad \dim_m X = \text{Ord } M_m(X).
$$

If $\dim_m X = \infty$, then we say that the dimension $\dim_m X$ is not defined. Comparing (3.1) and (3.2), we see that each of the functions $\dim_m$ is a transfinite extension of Lebesgue covering dimension.

**Theorem 4 ([8]).** For a compact space $X$, the dimension $\dim_m X$ is defined if and only if $X$ is an $m$-C-space.

For $m = 2$ and $m = \infty$, Theorem 4 was proved by Borst in [4, 6]. Clearly, if $m_1 \leq m_2$, then $\dim_{m_1} X \leq \dim_{m_2} X$.

**Question 23.** Does the equality $\dim_m = \dim_{m+1}$ hold in the class of compact metrizable spaces for all $m$?

In view of Theorem 4, a positive answer to Question 23 would give a positive answer to Question 2. For the compact space $E_{\omega_0}$ constructed by Borst in [5], we have $\dim_2 E_{\omega_0} = \omega_0 < \infty = \dim_\infty E_{\omega_0}$.

Using the ideas of R. Pol from [16], it is easy to prove the following theorem.

**Theorem 5 ([8]).** Let $\mathcal{E}$ be a family of $m$-C-compact spaces. Then there exists an $m$-C-compact space into which all compact spaces from $\mathcal{E}$ can be embedded if and only if $\sup \{ \dim_m X : X \in \mathcal{E} \} < \omega_1$.

Thus, to give a negative answer to Question 2, it is sufficient to construct a family of $(m+1)$-C-compact spaces $X_\alpha$, where $\alpha \in \omega_1$, such that $\sup \{ \dim_m X_\alpha : \alpha \in \omega_1 \} < \omega_1$ but $\sup \{ \dim_{m+1} X_\alpha : \alpha \in \omega_1 \} = \omega_1$.

**Question 24.** Does there exist a compact metrizable space $X$ such that $\dim_m X < \dim_\infty X$ for all integer $m$?

A negative answer to Question 3 would imply a positive answer to Question 24.

**Question 25.** Does there exist an infinite-dimensional metrizable $C$-compactum $X$ containing no subcompacta $Y$ of dimension $0 < \dim_\infty Y < \dim_\infty X$?

### 3.3. Transfinite dimension $\dim_{wm}$. For a normal space $X$, we denote the set of all $m$-systems in $X$ by $\varphi_m(X)$.

We set $L_m(X) = \{ \sigma \in \text{Fin } \varphi_m(X) : \sigma \text{ is essential} \}$. For a normal space $X$, we have $\dim X \leq n$ if and only if $\text{Ord } L_m(X) \leq n$. Thus, it is natural to define $\dim_{wm} X = \text{Ord } L_m(X)$. If $\dim_{wm} X = \infty$, then we say that the dimension $\dim_{wm} X$ is not defined.

The function $\dim_{wm}$, as well as $\dim_m$, is a transfinite extension of the covering dimension.
Theorem 6 ([8]). For a compact space $X$, the dimension $\dim_{wm} X$ is defined if and only if $X$ is a $w$-$m$-$C$-space.

Clearly, if $m_1 \leq m_2$ then $\dim_{wm_1} X \leq \dim_{wm_2} X$.

**Question 26.** Does the equality $\dim_{wm} = \dim_{w(m+1)}$ hold in the class of compact metrizable spaces for all $m$?

By virtue of Theorem 6, a positive answer to Question 26 gives a positive answer to Question 6.

**Question 27.** Does there exist a compact metrizable space $X$ such that $\dim_{w2} X < \dim_{w\infty} X$?

A negative answer to Question 7 implies a positive answer to Question 27.

**Proposition** ([8]). If $X$ is a normal space and $m \geq 3$ is an integer or $m = \infty$, then $\dim_{wm} X \leq \dim_m X$. Moreover, $\dim_{w2} X = \dim_2 X$.

**Question 28.** Does there exist a compact metrizable space $X$ such that $\dim_{w\infty} X < \dim_m X$ for some $m \geq 2$ or for $m = \infty$?

The Borst compact space $E_{\omega_0}$ gives a positive answer to one of Questions 27 and 28. Borst’s question of whether $\dim_2 (X \times I) = \dim_2 X + 1$ for any compact metrizable space $X$ can be generalized as follows.

**Question 29.** Let $X$ be a compact metrizable space. Is it true that

1. $\dim_m (X \times I) = \dim_m X + 1$;
2. $\dim_{wm} (X \times I) = \dim_{wm} X + 1$?

**3.4. Inductive dimensions.** Borst’s inequality $\dim_2 X \leq \Ind X$ [4, Theorem 3.2.4] can be strengthened as $\dim_{w\infty} X \leq \Ind X$.

**Question 30.** Does the inequality $\dim_{\infty} \leq \Ind$ hold in the class of all compact metrizable spaces?

This question has the following weak version.

**Question 31.** Does the inequality $\dim_m \leq \Ind$ hold in the class of all compact metrizable spaces for some integer $m \geq 3$?

A pair $(u, \Phi)$, where $u = \{U_1, \ldots, U_k\} \in \text{cov}_m (X)$ and $\Phi = \{F_1, \ldots, F_k\} \in \varphi_m (X)$, is called an $m$-covering pair if $F_i \subset U_i$ for each $i$. If $O\Phi = \{OF_1, \ldots, OF_k\}$ is a neighborhood of $\Phi$ refining $u$, then the set $P = X \setminus \bigcup O\Phi$ is called a partition of the covering pair $(u, \Phi)$.

**Definition.** The large transfinite inductive dimension $\Ind_m$ (where $m$ is an integer $\geq 2$ or $m = \infty$) in the class of all normal spaces is defined as follows:

1. $\Ind_m X = -1$ if and only if $X = \emptyset$;
2. $\Ind_m X \leq \alpha$, where $\alpha$ is an ordinal, if, for every $m$-covering pair $(u, \Phi)$, there exists a partition $P$ of $(u, \Phi)$ such that $\Ind_m P < \alpha$;
3. $\Ind_m X = \alpha$ if $\Ind_m X \leq \alpha$ and $\Ind_m X \leq \beta$ for no $\beta < \alpha$;
(d) $\text{Ind}_m X = \infty$ if $\text{Ind}_m X \leq \alpha$ for no ordinal $\alpha$.

For every normal space $X$, we have

$$\text{Ind} X = \text{Ind}_2 X \leq \cdots \leq \text{Ind}_m X \leq \text{Ind}_{m+1} X \leq \cdots \leq \text{Ind}_\infty X.$$ 

**Theorem 7** ([8]). The dimension $\text{Ind}_m$ is defined for any hereditarily normal compact space which can be represented as a countable union of subspaces for which the dimension $\text{Ind}_m$ is defined.

**Corollary.** For any countable-dimensional compact metrizable space, the dimensions $\text{Ind}_\infty$ and, therefore, $\text{Ind}_m$ for all $m$ are defined.

**Theorem 8** ([8]). If the dimension $\text{Ind}_m$ is defined for a compact space $X$ with weight $w(X) \leq \omega_\alpha$, then $\text{Ind}_m X \leq \omega_{\alpha+1}$.

**Theorem 9** ([8]). For any normal space $X$, $\text{dim}_m X \leq \text{Ind}_m X$.

**Question 32.** Does the equality $\text{Ind}_m = \text{Ind}_{m+1}$ hold in the class of all compact metrizable spaces for all $m$?

**Question 33.** Is it true that $\text{Ind} X = \text{Ind}_\infty X$ for an arbitrary compact metrizable space $X$?

**References**


Some problems in the dimension theory of compacta

Boris A. Pasynkov

Dedicated to the 70th anniversary of the publication of Alexandroff’s problem on the dimensions of compacta.

All topological spaces considered in this paper are assumed to be Tychonoff and called simply spaces; by maps we mean continuous maps of topological spaces.

Almost all problems posed below concern compact spaces. Recall that the dimension $\Delta$ of a paracompact space $X$ is defined as follows: $\Delta X \leq n$ if there exists a strongly zero-dimensional paracompact space $X^0$ and a surjective closed map $f: X^0 \to X$ such that $|f^{-1}x| \leq n + 1$ for any $x \in X$.

1. On the coincidence of $\text{dim}$, $\text{ind}$, $\text{Ind}$, and $\Delta$ for compact spaces

It is well known that the three basic dimension functions $\text{dim}$, $\text{ind}$, and $\text{Ind}$ coincide for compact metrizable spaces, i.e.,

\[ \text{dim} X = \text{ind} X = \text{Ind} X \]

for any compact metrizable space $X$. In 1936, Alexandroff [1] asked whether they coincide for arbitrary compact spaces. In 1941, he proved that $\text{dim} X \leq \text{ind} X$ for any compact space $X$. Recall also that $\text{ind} X \leq \text{Ind} X$ for any normal space $X$ and (see [22]) $\text{Ind} X \leq \Delta X$ for any paracompact space $X$. Moreover, for any metrizable space $X$, $\text{dim} \beta X = \text{ind} \beta X = \text{Ind} \beta X = \Delta X$ and there exists a compactification $cX$ of $X$ such that $\text{dim} cX = \text{ind} cX = \text{Ind} cX = \Delta cX$ and $w(cX) = w(X)$.

In 1958, Pasynkov proved that $\text{dim} G = \text{ind} G = \text{Ind} G$ for any compact group $G$; in 1962, he obtained the equalities $\text{dim} G/H = \text{ind} G/H = \text{Ind} G/H$ for any locally compact group $G$ and any closed subgroup $H$ of $G$ (in particular, they hold for compact coset spaces $G/H$). After that, the following definition and problem naturally arose.

A compact space $X$ is called algebraically homogeneous if there exists a topological group $G$ and its closed subgroup $H$ such that $X$ is homeomorphic to $G/H$.

Question 1. Do all or some of the dimensions $\text{dim}$, $\text{ind}$, and $\text{Ind}$ coincide for algebraically homogeneous compact spaces?

The following problem is related to Question 1.

Question 2. Describe the topological groups $G$, their closed subgroups $H$, and compact coset spaces $G/H$ for which all or some of the dimensions $\text{dim} G/H$, $\text{ind} G/H$, and $\text{Ind} G/H$ coincide. Do they coincide for first countable, perfectly normal, dyadic, hereditarily normal coset spaces?
Note that a zero-dimensional perfectly normal (and, hence, first countable) compact coset space $G/H$ may not be metrizable or dyadic (for example, Alexandroff’s double arrow space is neither metrizable nor dyadic).

Partial answers to Questions 1 and 1 were given by Pasynkov in [15, 17, 18]. He called a topological group $G$ almost metrizable if there exists a compact set $C \subset G$ and its neighborhoods $O_i$, where $i \in \mathbb{N}$, such that any neighborhood of $C$ is contained in $O_i$ for some $i$; a space $X$ is almost metrizable if there exists a compact group $G$ which acts continuously on $X$ so that the orbit space $X/G$ is metrizable (see [15, 17]). All groups of pointwise countable type (in particular, all paracompact $p$, Čech complete, and locally compact groups) are almost metrizable. It was proved in [17] that $\dim X = \text{Ind} X = \Delta X$ for any almost metrizable space $X$ and that if $\dim X < \infty$, then $X$ admits a perfect zero-dimensional (i.e., having zero-dimensional fibers) map $f$ onto a metrizable space. Thus, for any almost metrizable compact space $X$, relations $(\ast)$ hold, $\dim X = \Delta X$, and if $\dim X < \infty$, then $X$ admits a zero-dimensional map onto a metrizable compact space.

Since any coset space $G/H$ of a closed subgroup $H$ in an almost metrizable group $G$ is almost metrizable (see [18]), it follows that for any compact coset space $G/H$, where $G$ is an arbitrary almost metrizable group (and $H$ is its closed subgroup), $\dim G/H = \text{Ind} G/H = \Delta G/H$ and if $\dim G/H < \infty$, then $G/H$ admits a zero-dimensional map onto a compact metrizable space. On the other hand, if a compact space $X$ admits a zero-dimensional map onto a compact metrizable space, then relations $(\ast)$ hold and $\dim X = \Delta X$ (the equality $\dim X = \text{Ind} X$ was proved in [14] and $\dim X = \Delta X$ in [6, 17]; as far as I know, the inequality $\dim X = \text{Ind} X$ is due to Katětov). Note that if a compact space $X$ is a $G_\delta$-subset of a topological group and $\dim X < \infty$, then $X$ admits a zero-dimensional map onto a compact metrizable space (and hence relations $(\ast)$ hold).

It is not known whether there exist algebraically homogeneous compact spaces with noncoinciding dimensions. The situation with topologically homogeneous compacta is clearer.

In 1971, Fedorchuk [8] constructed a topologically homogeneous first countable compact space $F$ with $\dim F = 1 < \text{ind} F = 2$. Later, in 1990, Chatyrko [2] constructed first countable topologically homogeneous compact spaces $C_n$ with $\dim C_n = 1$ and $\text{ind} C_n = n$ for $n = 2, 3, \ldots$.

1379? Question 3. Do the dimensions $\text{ind}$ and $\text{Ind}$ coincide for any (first countable, hereditarily normal, hereditarily paracompact, dyadic, (hereditarily) separable) topologically homogeneous compact space?

1380? Question 4 ([20]). Does there exist a topologically homogeneous compact space $T_\alpha$ with $\dim T_\alpha < \infty$ and $\text{trind} X_\alpha = \alpha$ for any ordinal $\alpha \geq \omega_0$?

In this question, the topological homogeneity of $T_\alpha$ can be replaced by the weaker requirement that $\text{trind}_x T_\alpha = \alpha$ for all $x \in T_\alpha$ (dimensional homogeneity).
**Question 5.** Do there exist topologically homogeneous compact spaces that have infinite dimension dim and are weakly infinite-dimensional or weakly (≡ strongly (see [7])) countable-dimensional?

2. **Noncoincidence of dim and ind for compact spaces**

In 1949, Lunc constructed a compact space $L_1$ with $\dim L_1 = 1 < \ind L_1 = 2$. Then (also in 1949), Lokutsievskii constructed a compact space $L_2$ with the same dimensions dim and ind being the union of two compact subspaces with dimensions ind and dim equal to 1. In 1958, for any positive integers $n$ and $m > n$, Vopěnka constructed compact spaces $X_{mn}$ and $Y_{mn}$ such that $\dim X_{mn} = \ind Y_{mn} = m$, $\ind X_{mn} = n$, and $\dim Y_{mn} = n$. These results had completely clarified the relations between the dimensions dim and ind in the class of all compact spaces.

The first example of a first countable compactum with noncoinciding dimensions dim and ind was suggested by Fedorchuk in 1968. Then, in 1970, Filippov constructed first countable compacta $F_{mn}$ with $\dim F_{mn} = m$ and $\ind F_{mn} = n$ for any positive integers $m$ and $n > m$. General approaches to constructing compact spaces with noncoinciding dimensions dim and ind were suggested in [3, 4].

Recall that a compact space $X$ is Dugundji if $X$ is the limit of an inverse system $\{X_\alpha, p_\beta\alpha; \alpha \in A\}$ of compacta with the following properties: $A$ is the set of all ordinals $< \tau$ for some $\tau$; all of the bonding maps $p_\beta\alpha$ are open and surjective; $X_1$ is metrizable; there exists a metrizable space $M$ and maps $q_\alpha: X_{\alpha+1} \to M$ for all $\alpha, \alpha + 1 \in A$ such that the diagonal $p_\alpha + 1\alpha \Delta q_\alpha$ is a topological embedding; for any limit ordinal $\gamma \in A, X_\gamma$ is the limit of the system $\{X_\alpha, p_\beta\alpha; \alpha < \gamma\}$. Dugundji compacta are very close to topological products of compact metrizable spaces and have the following simple characterization: a compact space $X$ is Dugundji if and only if, for any zero-dimensional compact space $Z$, any closed subset $C$ of $Z$, and any map $f: C \to X$, there exists a map $F: X \to Z$ such that $F | C = f$.

In 1977, Fedorchuk constructed a Dugundji compactum $F$ of weight $\mathfrak{c}$ with $\dim F = 1$ and $\ind F = \Ind F = 2$. In 2002, Pasynkov and A.V. Odinokov constructed Dugundji compacta $PO_n$ with $\dim PO_n = 1$ and $\ind PO_n = n$. In [24], Uspenskij considered strongly homogeneous (≡ with rectifiable diagonal) compact spaces. They are Dugundji and homogeneous.

**Question 6.** Do the dimensions dim and ind coincide for homogeneous Dugundji compacta and for strongly homogeneous compacta?

A few years ago, the study of the dimensional properties of Eberlein compacta was initiated. I can construct strong Eberlein compacta (that is, compact subsets of the $\sigma$-products $\{x = \{x_\alpha\}_{\alpha \in A} \in \prod_{\alpha \in A} I_\alpha : |\{\alpha \in A : x_\alpha \neq 0\}| < \omega\}$ of unit intervals $I_\alpha = [0, 1]$) $P_n$ such that $\dim P_n = 1$ and $\ind P_n = n$ for $n = 2, 3, \ldots$.

**Question 7.** Do the dimensions dim and ind coincide for homogeneous (or hereditarily normal, hereditarily paracompact, first countable, perfectly normal) (strong) Eberlein (Corson, Valdivia) compacta?
3. On the noncoincidence of \( \text{ind} \) and \( \text{Ind} \) for compact spaces

In 1969, Filippov constructed a compactum \( F \) with \( \dim F = \text{ind} F = 2 < \text{Ind} F = 3 \) (the proofs were published in [11]). Then (in 1970 [10]), he explained (without detailed proofs) how to construct compacta \( F_i \) with \( \dim F_i = 1, \text{ind} F_i = \alpha_i, \) and \( \text{Ind} F_i = 2i - 1 \) for \( i = 2, 3, \ldots \). The following problem remains open.

**Question 8.** Do there exist a positive integer \( m \geq 2 \) and compact spaces \( A_{mn} \) for all integers \( n > m \) such that \( \dim A_{mn} = 1 \) and \( \text{ind} A_{mn} = m \) and \( \text{Ind} A_{mn} = n \)?

Of course, the most interesting case is \( m = 2 \).

If the answer to Question 8 is “yes”, then the one-point compactification \( A_{m\omega} \) of the discrete union of all \( A_{mn} \) has the properties \( \text{ind} A_{m\omega} = m \) and \( \text{trInd} A_{m\omega} = \omega \). Thus, the following question makes sense.

**Question 9.** Do there exist a positive integer \( m \geq 2 \) and compact spaces \( A_{m\alpha} \) for all transfinite numbers \( \alpha \) such that \( \text{ind} A_{m\alpha} = m \) and \( \text{trInd} A_{m\alpha} = \alpha \)?

I can construct a strong Eberlein compactum \( \Psi \) with \( \text{ind} \Psi = 2 \) and \( \text{Ind} \Psi = 3 \). So, Question 8 makes sense for (strong) Eberlein (Corson, Valdivia) compacta.

**Question 10.** Do the dimensions \( \text{ind} \) and \( \text{Ind} \) coincide for a (strong) Eberlein (Corson, Valdivia) compactum provided that it is first countable (homogeneous, hereditarily normal, hereditarily paracompact)?

Before Filippov’s results, it was known that \( \text{ind} X = \text{Ind} X \) for any perfectly normal compact space \( X \) (N. B. Vedenisov, 1939). Recall that a space \( X \) is said to be perfectly \( \kappa \)-normal [23] (quasi-perfectly normal [5]) if the closure of every open (respectively, \( G_\delta \)) subset of \( X \) is a zero-set. Obviously, any quasi-perfectly normal space is perfectly \( \kappa \)-normal, and any perfectly normal space is quasi-perfectly normal. In 1977, Fedorchuk [9] asked the following question.

**Question 11.** Is it true that \( \text{ind} X = \text{Ind} X \) holds for any perfectly \( \kappa \)-normal compact space \( X \)?

In 1982, Chigogidze [5] proved \( \text{ind} X = \text{Ind} X \) for any quasi-perfectly normal compact space \( X \). This gives a partial answer to Question 11, because any hereditarily normal quasi-perfectly normal space is perfectly \( \kappa \)-normal (this was proved by Chigogidze). Earlier (in 1977), Fedorchuk [9] proved that \( \text{ind} X = \text{Ind} X \) holds for any hereditarily perfectly \( \kappa \)-normal space \( X \) (hereditarily perfectly \( \kappa \)-normal means that every closed \( G_\delta \)-subset of \( X \) is perfectly \( \kappa \)-normal). In particular, \( \text{ind} X = \text{Ind} X \) holds for all Dugundji compacta and even for all \( \kappa \)-metrizable compacta. (Recall that a compact space \( X \) is \( \kappa \)-metrizable if \( X \) is the limit of a countably directed inverse system \( \{ X_\alpha, p_{\beta\alpha} : \alpha \in \mathcal{A} \} \) of compact metrizable spaces with surjective open bonding maps \( p_{\beta\alpha} \) such that, for any increasing sequence \( \alpha(i) \in \mathcal{A} \), where \( i \in \mathbb{N} \), the supremum \( \beta = \sup \{ \alpha(i) : i \in \mathbb{N} \} \) in \( \mathcal{A} \) is defined and \( X_\beta \) is the limit of the inverse sequence \( \{ X_{\alpha(i)}, p_{\alpha(i+1)\alpha(i)} : i \in \mathbb{N} \} \).)
4. Dimensional properties of topological products

We start with the following old problem.

**Question 12.** Is it true that \( \text{Ind} X \times I \leq \text{Ind} X + 1 \) for any (first countable) compact space \( X \)?

In 1972 \([12]\), Filippov constructed compact spaces \( X \) and \( Y \) such that \( \text{ind} X = \text{Ind} X = 1 \), \( \text{ind} Y = \text{Ind} Y = 2 \), and \( \text{ind} X + \text{ind} Y = \text{Ind} X + \text{Ind} Y = 3 < \text{ind} X \times Y \leq \text{Ind} X \times Y \). In 1999 (in his Ph.D. thesis), D. V. Malykhin strengthened this result of Filippov. He constructed compact spaces \( X \) and \( Y \) such that they have the same dimensional properties as those constructed by Filippov and \( X \) has the additional property of being linearly ordered. Malykhin conjectured that, instead of his compact space \( X \), the lexicographically ordered square can be taken.

**Question 13.** How large can the gap between \( \text{ind} X \times Y \) and \( \text{Ind} X \times Y \) be for compact spaces \( X \) and \( Y \) with (given) \( \text{ind} X = \text{Ind} X \) and \( \text{ind} Y = \text{Ind} Y \)? What if \( X \) and \( Y \) are first countable?

Question 13 is interesting for many special classes of compact spaces, including the classes of (strong) Eberlein, Corson, and Valdivia compacta.

In \([21]\), an integer-valued function \( f(k,l) \) for \( k,l = 0,1,2,\ldots \) with the following property was defined: for any compact spaces \( X \) and \( Y \) with finite \( \text{Ind} X \) and \( \text{Ind} Y \), we have \( \text{Ind} X \times Y \leq f(\text{Ind} X, \text{Ind} Y) \) (this implies, in particular, that \( X \times Y \) always has finite dimension \( \text{Ind} \) provided that both \( X \) and \( Y \) have finite dimension \( \text{Ind} \)).

**Question 14.** Is it possible to improve the estimate \( \text{Ind} X \times Y \leq f(\text{Ind} X, \text{Ind} Y) \)?

Note that \( \text{Ind} X \times Y \leq \text{Ind} X + \text{Ind} Y \) for any compact spaces \( X \) and \( Y \) satisfying the conditions of the finite-sum theorem for \( \text{Ind} \) \([16]\).

**Question 15.** Do there exist perfectly normal (Dugundji, \( \infty \)-metrizable, perfectly \( \infty \)-normal) compact spaces \( X \) and \( Y \) such that \( \dim X = \text{ind} X \), \( \dim Y = \text{ind} Y \), and \( \dim X \times Y < \text{ind} X \times Y \)? Can these relations hold if \( Y \) is metrizable or \( X \times Y \) is perfectly normal?

The next problem is about the dimensional properties of products of noncompact spaces; I believe, this is one of the most interesting problems concerning the dimensional properties of topological products.

**Question 16** (see \([13, 19]\)). Is it true that \( \dim X \times Y \leq \dim X + \dim Y \) if \( X \times Y \) is paracompact (and \( X \) and \( Y \) are strongly paracompact)?

5. A problem concerning the subset theorem

**Question 17.** Is it true that \( \dim A \leq \dim X \) for any metrizable subset \( A \) of a (strong) Eberlein (or Corson) compactum \( X \)?

Obviously, it can be assumed without loss of generality that \( A \) is dense in \( X \).
References


Problems from the Lviv topological seminar

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Introduction

This collection of problems is formulated by participants and guests of the Lviv topological seminar held at the Ivan Franko Lviv National University (Ukraine).

1. Asymptotic dimension

We recall that a metric space $X$ is proper if the distance $d(\cdot, x_0)$ to a fixed point is a proper map for any $x_0 \in X$. A map $f : X \to Y$ between metric spaces is called coarse if it satisfies the following two conditions [34]:

**Coarse Uniformity:** There is a monotone function $\lambda : [0, \infty) \to [0, \infty)$ such that $d_Y(f(x), f(x')) \leq \lambda(d_X(x, x'))$;

**Metric Properness:** The preimage $f^{-1}(B)$ is bounded for every bounded set $B \subset Y$.

Two maps $f, g$ into a metric space $Y$ are close if there exists a constant $C > 0$ such that $d_Y(f(x), g(x)) < C$, for every $C > 0$. Two metric spaces $X, Y$ are said to be coarse equivalent if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ such that the maps $gf$ and $1_X$ are close and also $fg$ and $1_Y$ are close.

For a proper metric space $X$ the Higson compactification $\bar{X}$ is defined by means of the following proximity: $A \delta B$ if and only if $\lim_{r \to \infty} d(A \setminus B_r(x_0), B \setminus B_r(x_0)) < \infty$ if $\text{diam } A = \text{diam } B = \infty$ and $d(A, B) = 0$ otherwise. Here $x_0 \in X$ is a base point, $B_r(x_0)$ is the $r$-ball centered at $x_0$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

The remainder $\nu X = \bar{X} \setminus X$ of the Higson compactification is called the Higson corona [34].

The asymptotic dimension $\text{asdim}$ of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups [25]. This dimension can be considered as an asymptotic analogue of the Lebesgue covering dimension $\dim$. Dranishnikov has introduced the dimension $\text{asInd}$ which is analogous to the large inductive dimension $\text{Ind}$ (see [19]). It is known that $\text{asdim } X = \text{asInd } X$ for each proper metric space with $\text{asdim } X < \infty$. The problem of coincidence of $\text{asdim}$ and $\text{asInd}$ is still open in the general case [19].

The addition theorem for $\text{asdim}$ is proved in [13]: suppose that a metric space $X$ is presented as a union $A \cup B$ of its subspaces. Then $\text{asdim } X \leq \max\{\text{asdim } A, \text{asdim } B\}$.

We have also a weaker result for the dimension $\text{asInd}$: let $X$ be a proper metric space and $X = Y \cup Z$ where $Y$ and $Z$ are unbounded sets. Then $\text{asInd } X \leq \text{asInd } Y + \text{asInd } Z$ (see [32]).

We do not know whether this estimate is the best possible.
Question 1.1. Let $X$ be a proper metric space and $X = Y \cup Z$. Is it true that $\text{asInd} X \leq \max\{\text{asInd} Y, \text{asInd} Z\}$?

Let us note that the negative answer to this question gives us a negative answer to the problem of coincidence of asymptotic dimensions.

Extending codomain of $\text{Ind}$ to ordinal numbers we obtain the transfinite extension $\text{trInd}$ of the dimension $\text{Ind}$. It is known that there exists a space $S_\alpha$ such that $\text{trInd} S_\alpha = \alpha$ for each countable ordinal number $\alpha$ \cite{22}. This method does not work for $\text{asInd}$: the extension $\text{trasInd}$ appears to be trivial: if $\text{trasInd} X < \infty$, then $\text{asInd} X < \infty$ (see \cite{32}). However there exists a nontrivial transfinite extension $\text{trasdim}$ of $\text{asdim}$ (see \cite{33}): there is a metric space $X$ with $\text{trasdim} X = \omega$.

Question 1.2. Find for each countable ordinal number $\xi$ a metric space $X_\xi$ with $\text{trasdim} X_\xi = \xi$.

In the classical dimension theory of infinite dimensional spaces there is a special class of spaces that have property C. Properties of such spaces are close to those of finite-dimensional spaces. Dranishnikov defined an asymptotic analogue of property C \cite{18}.

Question 1.3. Let $X$ and $Y$ be two metric spaces with the asymptotic property C. Does $X \times Y$ have the asymptotic property C?

It is known that the dimension $\text{trasdim}$ classifies the class of metric spaces with the asymptotic property C. Hence a positive answer to the following question gives us the positive answer to the Question 1.3.

Question 1.4. Is there a function $\alpha : \omega_1 \to \omega_1$ such that $\text{trasdim} X \times Y \leq \alpha(\xi)$ for each countable ordinal number $\xi$ and two metric spaces $X$, $Y$ with $\text{trasdim} X \leq \xi$ and $\text{trasdim} Y \leq \xi$?

Arhangel’skiï introduced the dimension $\text{Dind}$ (see \cite{21}). This dimension has an asymptotic counterpart.

For a proper metric space $(X, d)$ we let $\text{asDind} X = -1$ if and only if $X$ is bounded. Suppose that we have already defined the class of proper metric spaces for which $\text{asDind} X \leq n - 1$. We say that $\text{asDind} X \leq n$ if for every finite family $U$ of open in the Higson compactification $\bar{X}$ sets there exists a finite family $V$ of open subsets in $\bar{X}$ with the following property: the family $\{V \cap \nu X : V \in \mathcal{V}\}$ is a discrete in $\nu X$ family which refines $U$ and $\text{asDind} X \setminus \bigcup \mathcal{V} \leq n - 1$.

Question 1.5. Find relations between the dimension $\text{Dind}$ and the other asymptotic dimension functions.

It is proved in \cite{20} that every proper metric space of asymptotic dimension 0 is coarsely equivalent to an ultrametric space. Recall that a metric $d$ on a set $X$ is called an ultrametric if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for every $x, y, z \in X$. The mentioned results from \cite{20} is an asymptotic version of the classical de Groot’s result characterizing zero-dimensional metric spaces as those admitting a compatible ultrametric.
Nagata [30] introduced a counterpart of the notion of ultrametric: a metric $d$ on a set $X$ is said to satisfy property $(\ast)_n$ if, for every $x, y_1, \ldots, y_{n+2} \in X$, there exist $i, j$, $i \neq j$, such that $d(y_i, y_j) \leq d(x, y)$.

**Question 1.6.** Is every proper metric space $(X, d)$ with $\text{asdim } X \leq n$ coarsely equivalent to a proper metric space whose metric satisfies $(\ast)_n$?

There are another classes of metrics that characterize covering dimension (see, e.g., [27]).

**Question 1.7.** Are there metrics that characterize as above the asymptotic dimension $n \geq 1$?

2. Extension of metrics

The problem of existence of linear regular operators (i.e., operators of norm 1 that preserve linear combination with nonnegative coefficients), extending (pseudo)-metrics was formulated by C. Bessaga [14] and solved by T. Banakh [2].

**Question 2.1.** Is there a linear operator that extends metrics from a compact metrizable space $X$ to left invariant metrics on a free topological group of $X$?

A similar question can be formulated for extension of metrics from a compact metrizable space $X$ to norms on the free linear space over $X$.

Let $(X, d)$ be a compact metric space. Given a subset $A$ of $X$, we say that a pseudometric $\varrho$ on $A$ is Lipschitz if there is $C > 0$ such that $d(x, y) \leq C \varrho(x, y)$, for any $x, y \in A$. Also, a function $f : A \to \mathbb{R}$ is Lipschitz if there is $C > 0$ such that $|f(x) - f(y)| \leq Cd(x, y)$, for every $x, y \in A$. Denote by lpm$(A)$ (resp. lpf$(A)$) the set of all Lipschitz pseudometrics (resp. functions) on $A$. The set lpm$(A)$ (resp. lpf$(A)$) is a cone (resp. linear space) with respect to the operations of pointwise addition and multiplication by scalar. We endow lpm$(A)$ with the norm $\|\cdot\|_{\text{lpm}(A)}$:

$$\|\varrho\|_{\text{lpm}(A)} = \sup \left\{ \frac{\varrho(x, y)}{d(x, y)} : x \neq y \right\}$$

and lpf$(A)$ with the seminorm $\|\cdot\|_{\text{lpf}(A)}$:

$$\|f\|_{\text{lpf}(A)} = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x \neq y \right\}.$$ 

We say that a map $u : \text{lpm}(A) \to \text{lpm}(X)$ is an extension operator for Lipschitz pseudometrics if the following holds:

1. $u$ is linear (i.e., $u(\varrho_1 + \varrho_2) = u(\varrho_1) + u(\varrho_2)$, $u(\lambda \varrho) = \lambda u(\varrho)$ for every $\varrho, \varrho_1, \varrho_2 \in \text{lpm}(A)$, $\lambda \in \mathbb{R}_+$);
2. $u(\varrho)|_{(A \times A)} = \varrho$, for every $\varrho \in \text{lpm}(A)$;
3. $u$ is continuous in the sense that $\|u\| = \sup\{\|u(\varrho)\|_{\text{lpm}(X)} : \|\varrho\|_{\text{lpm}(A)} \leq 1\}$ is finite.

This definition is a natural counterpart of those introduced in [15] for the extensions of Lipschitz functions. The following notation is introduced in [15]:

$$\lambda(A, X) = \inf\{\|u\| : u \text{ is a linear extension operator from lpf}(A) \text{ to lpf}(X)\}.$$
Similarly, we put
\[ \Lambda(A, X) = \inf \{ \|u\| : u \text{ is a linear extension operator from } \text{lpm}(A) \text{ to } \text{lpm}(X) \} . \]

In [15] the problem of existence of extension operators of Lipschitz functions is considered. It is natural to formulate the corresponding problem for pseudometrics.

**Question 2.2.** Let \( A \) be a closed subspace of a compact metric space \( X \). Is there an extension operator for Lipschitz pseudometrics \( u : \text{lpm}(A) \rightarrow \text{lpm}(X) \)?

**Question 2.3.** Compare \( \Lambda(S, X) \) and \( \lambda(S, X) \).

### 3. Questions in general topology

All topological spaces in this section are assumed to be Hausdorff, see [5, 6] for undefined notions used below.

**Question 3.1.** Is there an interplay between topological properties of a compact topological inverse semigroup \( S \) and those of the maximal Clifford semigroup \( C \subset S \) and the maximal sublattice \( E \)? In particular:

(a) Is \( S \) countably cellular (or separable) if so is the space \( C \)?
(b) Is \( S \) countably cellular if the maximal semilattice \( E \) is second countable?
(c) Is \( S \) (hereditary) separable if all maximal groups of \( S \) are (hereditary) separable and the maximal semilattice is Lawson and (hereditary) separable?
(d) Is \( S \) fragmentable (resp. Corson, Eberlein, Gul'ko, Radon–Nikodým, or Rosenthal) compact if so is the Clifford semigroup \( C \)?

By a mean on a space \( X \) we understand any commutative idempotent operation \( m : X \times X \rightarrow X \). Associative means are also called semilattice operations. Each scattered metrizable compact space, being homeomorphic to an ordinal interval \([0, \alpha]\), admits a continuous associative mean (just take the operations \( \min \) or \( \max \) on \([0, \alpha]\)).

**Question 3.2.** Does any scattered compact Hausdorff space \( X \) admit a (separately) continuous mean?

It should be noted that there exist scattered compact Hausdorff spaces admitting no separately continuous associative mean, see [7].

Our other question is due to V. Maslyuchenko, V. Mykhaylyuk and O. Sobchuk and relates to the classical theorem of Baire on functions of the first Baire class. We recall that a function \( f : X \rightarrow Y \) between topological spaces is called

- **of the first Baire class** if \( f \) is the pointwise limit of a sequence of continuous functions;
- **\( F_\alpha \)-measurable** if the preimage \( f^{-1}(U) \) of any open set \( U \subset Y \) is of type \( F_\alpha \) in \( X \).
It is well-known that each function $f : X \to Y$ of the first Baire class with values in a perfectly normal space is $F_\sigma$-measurable. The converse is true if $X$ is metrizable and the space $Y$ is metrizable, separable, connected and locally path-connected, see [24, 37].

**Question 3.3.** Is each $F_\sigma$-measurable function $f : [0, 1] \to C_p[0, 1]$ a function of the first Baire class?

This question is equivalent to the original question of V. Maslyuchenko, V. Mykhaylyuk, and O. Sobchuk [29]:

**Question 3.4.** Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be a function continuous with respect to the first variable and of the first Baire class with respect to the second variable. Is $f$ the pointwise limit of separately continuous functions?

Let $P$ be a property of a subset in a topological space. A topological space is called an $AP$-space (resp. $WAP$-space) if for every subset $B \subset X$ and every (resp. some) point $x \in \overline{B} \setminus B$ there exists a subset $C \subset B$ with the property $P$ in $X$ such that $x \in C$.

For example, a space $X$ has countable tightness if it is an $AP$-space for the property $P$ of being a countable subset. A space $X$ is Fréchet–Urysohn (resp. sequential) if and only if $X$ is an $AP$-space (resp. $WAP$-space) where $P$ is the property of a subset $A \subset X$ to have compact metrizable closure. A space $X$ is a $k'$-space in the sense of Arhangel’skii [1] if and only if $X$ is an $\overline{A}$-$C$-space where $\overline{C}$ is the property of a subset $A \subset X$ to have compact closure in $X$.

**Question 3.5.** Find an example of a countably compact $WAC$-space which is not an $\overline{A}$-$C$-space.

Let $D$ (resp. $M$) denote the properties of a subspace to be discrete (resp. metrizable).

**Question 3.6.** Is every topological group of countable tightness an $AM$-space? $AD$-space?

**Question 3.7.** Let $X$ be an $AM$-space. Is the free topological group of $X$ an $AM$-space?

**Question 3.8.** Characterize the class of monothetic $AM$-groups ($AM$-paratopological groups)?

**Question 3.9.** Is every countable regular space an $AM$-space?

4. Some problems in Ramsey theory

In this section we ask some problems on symmetric subsets in colorings of groups. By an $r$-coloring of a set $X$ we understand any map $\chi : X \to \{1, \ldots, r\}$, which can be identified with a partition $X = \bigcup_r X_i$ of $X$ into $r$ disjoint pieces $X_i = \chi^{-1}(i)$. As a motivation for subsequent questions let us mention the following result of T. Banakh [3].
Theorem C. For any $n$-coloring of the group $\mathbb{Z}^n$ there is an infinite monochromatic subset $S \subset \mathbb{Z}^n$ symmetric with respect to some point $c \in \{0,1\}^n$.

A subset $S$ of a group $G$ is called symmetric with respect to a point $c \in G$ if $S = cS^{-1}c$.

This theorem suggests to introduce the cardinal function $\nu(G)$ assigning to each group $G$ the smallest cardinal number $r$ of colors for which there is an $r$-coloring of $G$ without infinite monochromatic symmetric subsets.

In [8] the value $\nu(G)$ was calculated for any abelian group $G$:

$$
\nu(G) = \begin{cases} 
  r_0(G) + 1 & \text{if } G \text{ is finitely generated} \\
  r_0(G) + 2 & \text{if } G \text{ is infinitely generated and } |G[2]| < \aleph_0 \\
  \max\{|G[2]|, \log |G|\} & \text{if } |G[2]| \geq \aleph_0 
\end{cases}
$$

where $r_0(G)$ is the free rank of $G$ and $G[2] = \{x \in G : 2x = 0\}$ is the Boolean subgroup of $G$.

Much less is known for non-commutative groups.

Question 4.1. Investigate the cardinal $\nu(G)$ for non-commutative groups $G$. In particular, is $\nu(F_2)$ finite for the free group $F_2$ with two generators?

The only information on $\nu(F_2)$ is that $\nu(F_2) > 2$, see [26].

Question 4.2. Has each finite coloring of an infinite group $G$ a monochromatic symmetric subset $S \subset G$ of arbitrarily large finite size? (The answer is affirmative if $G$ is Abelian.)

For every uncountable abelian group $G$ with $|G[2]| < |G|$ there is a 2-coloring of $G$ without symmetric monochromatic subsets of size $|G|$, see [31].

Question 4.3. Is it true that for every 2-coloring of an uncountable abelian group $G$ with $|G[2]| < |G|$ and for every cardinal $\kappa < |G|$ there is a monochromatic symmetric subset $S \subset G$ of size $|S| \geq \kappa$? (The answer is affirmative under GCH, see [26].)

There is another interesting concept suggested by Theorem C on colorings of the group $\mathbb{Z}^n$. Let us define a subset $C \subset \mathbb{Z}^n$ to be central if for any $n$-coloring of $\mathbb{Z}^n$ there is an infinite monochromatic subset $S \subset \mathbb{Z}^n$ symmetric with respect to a point $c \in C$. A central set $C \subset \mathbb{Z}^n$ is called minimal if it does not lie in any smaller central set.

Question 4.4. Describe the geometric structure of (minimal) central subsets of $\mathbb{Z}^n$. Is each minimal central subset of $\mathbb{Z}^n$ finite? What is the smallest size $c(\mathbb{Z}^n)$ of a central set in $\mathbb{Z}^n$?

It was proved in [4] that $\frac{n(n+1)}{2} \leq c(\mathbb{Z}^n) < 2^n$ and $c(\mathbb{Z}^n) = \frac{n(n+1)}{2}$ for $n \leq 3$.

Question 4.5. Calculate the number $c(\mathbb{Z}^4)$. (It is known that $12 \leq c(\mathbb{Z}^4) \leq 14$, see [4].)
Concerning the first (geometric) part of Question 4.4 the following information is available for small $n$, see [4]:

(1) a subset $C \subset \mathbb{Z}$ is central if and only if $C$ contains a point;
(2) a subset $C \subset \mathbb{Z}^2$ is central if and only if it contains a triangle $\{a, b, c\} \subset C$ (by which we understand a three-element affinely independent subset of $\mathbb{Z}^2$);
(3) each central subset $C \subset \mathbb{Z}^3$ of size $|C| = c(\mathbb{Z}^3) = 6$ is an octahedron $\{c \pm e_i : i \in \{1, 2, 3\}\}$ where $c \in \mathbb{Z}^n$ and $e_1, e_2, e_3 \in \mathbb{Z}^n$ are linearly independent vectors;
(4) there is a minimal central subset $C \subset \mathbb{Z}^3$ of size $|C| > 6$ containing no octahedron.

There is another numerical invariant $\text{ms}(X, S, r)$ related to colorings and defined for any space $X$ endowed with a probability measure $\mu$ and a family $S$ of measurable sets called symmetric subsets of $X$. By definition, $\text{ms}(X, S, r) = \inf \{\varepsilon > 0 : \text{for every measurable } r\text{-coloring of } X \text{ there is a monochromatic subset } S \in S \text{ of measure } \mu(S) \geq \varepsilon\}$. The notation “ms” reads as the maximal measure of a monochromatic symmetric subset and was suggested by Ya. Vorobets. If the family $S$ is clear from the context (as it is in case of groups), then we shall write $\text{ms}(X, r)$ instead of $\text{ms}(X, S, r)$.

The numerical invariant $\text{ms}(X, r)$ is defined for many natural algebraic and geometric objects: compact topological groups, spheres, balls etc. For such objects, typically, $\text{ms}(X, r)$ is equal to $\frac{1}{r}$, see [9, 11]. For example, in the case of the ball $B^n$ of the unit volume in the Euclidean space $\mathbb{R}^n$ of dimension $n \geq 2$ we get $\text{ms}(B^n, S, r) = \frac{1}{2r}$ for any family $S$ with $S_0 \subset S \subset S_+$ where $S_+$ is the family of measurable subsets of $B^n$ that are symmetric with respect to some non-trivial isometry of $\mathbb{R}^n$ and $S_0$ is the family of measurable subsets of $B^n$, symmetric with respect to some hyperplane passing through the center of the ball.

Moreover, for any measurable $r$-coloring of the ball $B^n$ of dimension $n \geq 3$ there is a monochromatic subset $S \in S_0$ of measure $\geq \frac{1}{2r}$. This phenomenon does not hold in dimension 2: there is a 2-coloring of the two-dimensional disk $B^2$ such that all monochromatic symmetric subsets $S \in S_0$ of $B^2$ have measure $\leq \frac{1}{4}$ (such an extremal coloring of the disk resembles the Chinese philosophical symbol “in-jan”, see [11]). The situation with the 1-dimensional ball $[0, 1]$ is even worse: we know that $\frac{1}{r^2 + r\sqrt{r^2 - 2}} \leq \text{ms}([0, 1], S_+, r) < \frac{1}{r^2}$ for $r > 1$ but the exact value of $\text{ms}([0, 1], S_+, r)$ is not known even for $r = 2$. However, we have some lower and upper bounds: $\frac{1}{4 + \sqrt{6}} \leq \text{ms}([0, 1], S_+, 2) < \frac{5}{24}$, see [11, 28] for more information. Observe that for $n = 1$ the family $S_+$ coincides with the family of subsets of $[0, 1]$, symmetric with respect to some point of $[0, 1]$. So, we shall write $\text{ms}([0, 1], r)$ instead of $\text{ms}([0, 1], S_+, r)$.

**Question 4.6.** Calculate the value $\text{ms}([0, 1], r)$, at least for $r = 2$. Can $\text{ms}([0, 1], 2)$ be expressed via some known mathematical constants?

It was proved in [11] that the limit $\lim_{r \to \infty} r^2 \cdot \text{ms}([0, 1], r)$ exists and lies in the interval $[\frac{1}{2}, \frac{1}{3}]$. 


Questions 4.7. Calculate the constant \( c = \lim_{r \to \infty} r^2 \cdot \text{ms}([0, 1], r) \).

More detail information on these problems can be found in the surveys [9, 10].

5. Questions on functors in the category of compact Hausdorff spaces

We denote by \( \text{Comp} \) the category of compact Hausdorff spaces and continuous maps. In the sequel, all the functors are assumed to be covariant endofunctors in \( \text{Comp} \).

First, we mention few examples of functors. The \textit{hyperspace functor} \( \exp \) assigns to every compact Hausdorff space \( X \) the set \( \exp X \) of nonempty closed subsets in \( X \) endowed with the Vietoris topology. A base of this topology consists of the sets of the form \( \{ A \in \exp X : A \subset \bigcup_i U_i, (\forall i) A \cap U_i \neq \emptyset \} \), where \( U_1, \ldots, U_n \) are open subsets in \( X \). Given a map \( f : X \to Y \) in \( \text{Comp} \), the map \( \exp f : \exp X \to \exp Y \) is defined by \( \exp f (A) = f(A) \).

The \textit{probability measure functor} \( P \) assigns to every compact Hausdorff space \( X \) the set \( P(X) \) of probability measures endowed with the weak* topology.

Let \( G \) be a subgroup of the permutation group \( S_n \). The \textit{\( G \)-symmetric power} of \( X \), \( \text{SP}^n_G(X) \), is the quotient space of \( X^n \) with respect to the natural action of \( G \) on \( X^n \) by permutation of coordinates. One can easily see that this construction determines a functor.

Some properties of the mentioned functors and other known functors were used by E.V. Shchepin [35] in order to introduce the notion of a normal functor. It turned out that normal functors, and some close to normal functors, found important applications in the topology of nonmetrizable compact Hausdorff spaces and other areas of topology (see, e.g., [23, 35, 36]).

If a functor \( F \) preserves embeddings, then, for a compact Hausdorff space \( X \) and a closed subspace \( A \) of \( X \), we always identify the space \( F(A) \) with a subspace in \( F(X) \) along the embedding \( F(i) \), where \( i : A \to X \) is the inclusion map.

Let a functor \( F \) preserve embeddings. We say that \( F \) \textit{preserves preimages} if \( F (f^{-1}(A)) = F(f)^{-1}(F(A)) \) for every map \( f : X \to Y \) and every closed subset \( A \) of \( Y \). We say that \( F \) \textit{preserves intersections} whenever \( F (\bigcap_{\alpha \in \Gamma} A_{\alpha}) = \bigcap_{\alpha \in \Gamma} F(A_{\alpha}) \) for every family of closed subsets \( \{ A_{\alpha} : \alpha \in \Gamma \} \) in \( X \).

An endofunctor \( F \) in \( \text{Comp} \) is called \textit{normal} (Shchepin [35]) if \( F \) preserves embeddings, surjections, weight of infinite compacta, intersections, preimages, singletons, the empty set, and the limits of inverse systems \( S = \{ X_\alpha, p_{\alpha\beta} : A \} \) over directed sets \( A \). More precisely, the latter condition means that the map \( h = (F(p_\alpha))_{\alpha \in A} \) is a homeomorphism of \( F(\lim S) \) onto \( \lim F(S) \), where \( p_\alpha : \lim S \to X_\alpha \) is the limit projection.

A functor \( F \) is said to be \textit{weakly normal} (almost normal) if it satisfies all the properties from the previous definition except perhaps the property of being epimorphic (respectively, the preimage preserving property).

The hyperspace functor \( \exp \), the probability measure functor \( P \), and the \( G \)-symmetric power functor \( \text{SP}^n_G \) are examples of normal functors.

For a functor \( F \) and a compact Hausdorff space \( X \) denote by \( F_n(X) \) the subspace \( \bigcup \{ F(f) (F(n)) : f \in C(n, X) \} \) of \( F(X) \) (here \( C(n, X) \) denotes the set of
all maps from the discrete space $n$ to $X$). Clearly, such a construction determines a subfunctor $F_n$ of $F$. A functor $F$ is of finite degree if there exists $n \in \mathbb{N}$ such that $F = F_n$.

If $\varphi = (\varphi_X): F \to F'$ is a natural transformation of functors then we say that $F$ is a subfunctor of $F'$ if all the components of $\varphi$ are inclusion maps and we say that $F'$ is a quotient functor of $F$ if all the components of $\varphi$ are onto maps.

The characteristic map of a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & \xrightarrow{u} & T
\end{array}
$$

in the category $\text{Comp}$ is the map $\chi: X \to Y \times_T Z = \{(y, z) \in Y \times Z : h(y) = u(z)\}$ defined by the formula $\chi(x) = (f(x), g(x))$. A diagram is bicommutative if its characteristic map is onto. A diagram is open-bicommutative if its characteristic map is open and onto.

A functor $F: \text{Comp} \to \text{Comp}$ is said to be bicommutative (resp. open-bicommutative) if $F$ preserves the class of bicommutative (resp. open-bicommutative) diagrams.

A functor is open if it preserves the class of open surjective maps. E.V. Shchepin proved that every open functor is bicommutative.

**Question 5.1** (Shchepin). Is every normal bicommutative functor open? 1424?

This problem was formulated more than 25 years ago. The notions of open and bicommutative functors were introduced by E.V. Shchepin [35]. The problem was solved in [40] for normal functors of finite degree.

**Question 5.2.** Is every normal bicommutative (open) functor open-bicommutative? 1425?

It is proved in [36] that natural transformations of (weakly, almost) normal functors form a set and therefore one can introduce the category of normal functors and their natural transformations.

A (weakly, almost) normal functor $F$ is called universal if every (weakly, almost) normal functor is isomorphic to a subfunctor of $F$.

**Question 5.3.** Is there a universal (weakly, almost) normal functor? 1426?

A normal functor $F$ is called couniversal if every normal functor $F'$ is a quotient functor of $F$.

**Question 5.4.** Is there a couniversal (weakly, almost) normal functor? 1427?

A normal functor $F$ is called zero-dimensional if $\dim F(X) = 0$ for every compact Hausdorff space $X$ with $\dim X = 0$.

**Question 5.5.** Is every normal functor a quotient functor of a zero-dimensional normal functor? 1428?
Let $\tau > \omega$ be a cardinal number. A functor $F$ is called $\tau$-normal if $F$ satisfies all the properties from the definition of normality except the preserving of weight and, in addition, the weight of $F(X)$ is $\leq \tau$, for every compact metrizable $X$ (the minimal $\tau$ for which a functor $F$ is $\tau$-normal is called the weight of $F$).

Actually, one can find the prototype of the notion of $\tau$-normal functor in Shchepin’s paper [35] as he considered the so-called normal functor-powers, i.e., the spaces of the form $F(X^\tau)$.

We say that a map $f : X \to Y$ satisfies the homeomorphism-lifting property if, for every homeomorphism $h : Y \to Y$ there exists a homeomorphism $h' : X \to X$ such that $fh' = hf$.

**Question 5.6** (Shchepin). Let $X$ be a metric compact space and $F$ a normal functor. Does the map $F((pr)^\tau) : F((X \times X)^\tau) \to F(X^\tau)$ satisfy the homeomorphism-lifting property?

**Question 5.7.** Is every multiplicative $\tau$-normal functor isomorphic to the power functor $(\cdot)^\tau$?

For normal functors, this problem was posed by Shchepin and solved in [38]. Shchepin proved the so-called spectral theorem, which states that, under some reasonable conditions, if a nonmetrizable compact Hausdorff space is represented as the inverse limit of two systems consisting of spaces of smaller weight then these systems contain isomorphic cofinal subsystems (see [35] for details). One can consider representations of $\tau$-normal functors as the limits of inverse systems consisting of functors of smaller weight and their natural transformations.

**Question 5.8.** Is there a counterpart of Shchepin’s spectral theorem in the category of $\tau$-normal functors?

Of special interest are functors of finite degree that preserve the class of compact metric ANR spaces (i.e., absolute neighborhood retracts). Basmanov [12] established such a property for a wide enough class of functors. Such functors are known to preserve other classes of spaces too: $Q$-manifolds (i.e., manifolds modeled on the Hilbert cube $Q = [0,1]^\omega$) [23], $n$-movable spaces [39], compact metric absolute neighborhood extensors in dimension $n$ [16].

We are going to formulate a few questions on the preservation of some geometric properties by functors of finite degree.

Let $P$ be a CW-complex. For any compact metric space $X$ the Kuratowski notation $X \tau P$ means the following: for every continuous map $f : A \to P$ defined on a closed subset $A$ of $X$ there is a continuous extension of $f$ onto $X$.

Denote by $L$ the class of all countable CW-complexes. Following [17], we define a preorder relation $\leq$ on $L$. For $L_1, L_2 \in L$, we have $L_1 \leq L_2$ if and only if $X \tau L_1$ implies $X \tau L_2$ for all compact metric spaces $X$. This preorder relation determines the following equivalence relation $\sim$ on $L$: $L_1 \sim L_2$ if and only if $L_1 \leq L_2$ and $L_2 \leq L_1$. We denote by $[L]$ the equivalence class containing $L \in L$.

For a compact metric space $X$, we say that its extension dimension does not exceed $[L]$ (briefly ext-dim $X \leq [L]$) whenever $X \tau L$. 
A compact metric space \( X \) is said to be an absolute (neighborhood) extensor in extension dimension \([L]\) if for any compact metric pair \((A, B)\) with \(\text{ext-dim } A \leq [L]\) and any continuous map \(f: B \to X\) there exists a continuous extension \(\bar{f}: A \to X\) (respectively \(\bar{f}: U \to X\), where \(U\) is a neighborhood of \(B\) in \(A\)) of \(f\).

In the sequel, we suppose that \(F\) is a normal functor of finite degree that preserves the class of compact metrizable ANR-spaces.

**Question 5.9.** Does \(F\) preserve the class of absolute (neighborhood) extensors in extension dimension \([L]\)?

Two maps \(f_0, f_1: X \to Y\) are said to be \([L]\)-homotopic if there exists a space \(Z\) with \(\text{ext-dim } Z \leq [L]\), a map \(\alpha: Z \to X \times [0, 1]\) which is \([L]\)-invertible (i.e., satisfies the property of lifting of maps from spaces of extension dimension \(\leq [L]\)), and a map \(H: Z \to Y\) such that \(f_i \alpha(z) = H(z)\), for every \(z \in \alpha^{-1}(X \times \{i\})\), \(i = \{0, 1\}\).

**Question 5.10.** Does \(F\) preserve the relation of \([L]\)-homotopy of maps?

We finish with the following question.

**Question 5.11.** Does \(F\) preserve the class of essential \(Q\)-\(M\)-factors, i.e., the class of spaces \(X\) such that \(X \times A\) is a \(Q\)-manifold for some \(A\) with \(\dim A < \infty\)?

**References**


Cantor set problems

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Introduction

A Cantor set is characterized as a topological space that is totally disconnected, perfect, compact and metric. Any two such spaces $C_1$ and $C_2$ are homeomorphic, but if $C_1$ and $C_2$ are subspaces of $\mathbb{R}^n$, $n \geq 3$, there may not be a homeomorphism of $\mathbb{R}^n$ to itself taking $C_1$ to $C_2$. In this case, $C_1$ and $C_2$ are said to be inequivalent embeddings of the Cantor set. There has been recent renewed attention to properties of embeddings of Cantor sets since these sets arise in the settings of dynamical systems, ergodic theory and group actions. The bibliography, while not complete, gives a sampling of the various mathematical areas where Cantor sets naturally arise.

A Cantor set $C$ in $\mathbb{R}^n$ is tame if it is equivalent to the standard middle thirds Cantor set. If it is not tame, it is wild. A Cantor set $C$ is strongly homogeneously embedded in $\mathbb{R}^n$ if every self homeomorphism of $C$ extends to a self homeomorphism of $\mathbb{R}^n$. At the opposite extreme, a Cantor set $C$ in $\mathbb{R}^n$ is rigidly embedded if the identity homeomorphism is the only self homeomorphism of $C$ that extends to a homeomorphism of $\mathbb{R}^n$. A Cantor set $C$ in $\mathbb{R}^n$ is slippery if for each Cantor set $D$ in $\mathbb{R}^n$ and for each $\epsilon > 0$, there is a homeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$, within $\epsilon$ of the identity, with $h(C) \cap D = \emptyset$.

ˇZeljko [28] defines the genus of a Cantor set $X$ in $\mathbb{R}^3$ and the local genus of points in $X$. A defining sequence for a Cantor set $X \subset \mathbb{R}^n$ is a sequence $(M_i)$ of compact $n$-manifolds with boundary such that $M_{i+1} \subset \text{int} M_i$ and $X = \bigcap_i M_i$. Let $\mathcal{D}(X)$ be the set of all defining sequences for $X$. For a disjoint union of handlebodies $M = \bigsqcup_{\lambda \in \Lambda} M_{\lambda}$, we define $g(M) = \sup \{\text{genus}(M_{\lambda}) : \lambda \in \Lambda\}$.

For any subset $A \subset X$, and for $(M_i) \in \mathcal{D}(X)$ we denote by $M_i^A$ the union of those components of $M_i$ which intersect $A$. The genus of the Cantor set $X$ with respect to the subset $A$, $g_A(X) = \inf \{g_A(X; (M_i)) : (M_i) \in \mathcal{D}(X)\}$, where $g_A(X; (M_i)) = \sup \{g(M_i^A) : i \geq 0\}$. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ the local genus of the Cantor set $X$ at the point $x$ and denote it by $g_x(X)$. For $A = X$ we call the number $g_X(X)$ the genus of the Cantor set $X$ and denote it by $g(X)$.

The problems

Antoine [2] produced the first example of a wild Cantor set in $\mathbb{R}^3$, the well-known Antoine’s necklace. Blankinship [6] extended Antoine’s construction to

The first author was supported in part by NSF grants DMS 0139678, DMS 0104325 and DMS 0453304. The second author was supported in part by Slovenian Research Agency program P1-0292-0101-04. Both authors were supported in part by Slovenian Research Agency grant SLO-US 2002/01 and BI-US/04-05/35.
higher dimensions, producing wild Cantor sets in Euclidean spaces of dimensions \( \geq 4 \). Daverman [8] produced an example of a strongly homogeneously embedded Cantor set if \( \mathbb{R}^n \) for \( n \geq 5 \). His example relied on decomposition theory results that only applied in high dimensions and on the existence of non simply connected homology spheres in dimensions \( \geq 3 \).

1447 **Question 1.** Is there a strongly homogeneously embedded wild Cantor set in \( \mathbb{R}^3 \) or \( \mathbb{R}^4 \), or are such sets necessarily tame?

The Antoine construction can be carefully done with sufficiently many tori at each stage so as to produce wild Cantor sets that are geometrically self similar and are Lipschitz homogeneously embedded in \( \mathbb{R}^3 \). See [12, 15, 29] for definitions and details. It is not clear that the Blankinship construction in higher dimensions can be done so as to produce geometrically self similar Cantor sets.

1447 **Question 2.** Is there a geometrically self similar wild Cantor set in \( \mathbb{R}^4 \) or in higher dimensions?

1447 **Question 3.** Are there Lipschitz homogeneously embedded wild Cantor sets in \( \mathbb{R}^4 \) or in higher dimensions?

Rushing [18] produced examples in \( \mathbb{R}^3 \) of wild Cantor sets of each possible Hausdorff dimension. At the end of his paper, he stated that a modification of the Blankinship construction would allow similar results in higher dimensions. Because of the difficulty in producing a self similar Blankinship construction, it is not clear how the generalization to higher dimensions would work.

1447 **Question 4.** Are there wild Cantor sets in \( \mathbb{R}^n \), \( n \geq 4 \) of arbitrary possible Hausdorff dimension?

DeGryse and Osborne [11] produced an example of a wild Cantor set in \( \mathbb{R}^3 \) with simply connected complement. Later, Skora [20] produced such Cantor sets using a different construction. Rigid wild Cantor sets in \( \mathbb{R}^3 \) and in higher dimensions were produced by Wright [24] using variations on the Antoine and Blankinship constructions. Garity, Repovš, and Željko [13] recently produced examples of rigid wild Cantor sets in \( \mathbb{R}^3 \) that also had simply connected complement. However the latter examples necessarily used tori of arbitrarily high genus in the construction.

1447 **Question 5.** Is there a rigid Cantor set in \( \mathbb{R}^3 \) with simply connected complement that has local genus \( n \) or less at every point, for some fixed \( n \)?

Bing–Whitehead Cantor sets are a generalization of the Cantor sets produced by DeGryse and Osborne. Ancel and Starbird [1] and later Wright [26] characterized which Bing–Whitehead constructions actually yield Cantor sets.

1447 **Question 6.** Is there a modification of the Bing–Whitehead Cantor set construction that yields rigid Cantor sets with simply connected complements?

1450 **Question 7.** Are Bing–Whitehead Cantor sets with infinite differences in the number of Whitehead constructions inequivalently embedded?
Sher in [19] showed that two equivalent Antoine Cantor sets necessarily had the same number of components in each stage of their defining sequences. In [12], the authors and Željko show that Antoine Cantor sets with the same number of components at each stage can be inequivalent. Knot theory techniques are used in the proof. This leads to the following question.

**Question 8.** Is it possible to completely classify Antoine Cantor sets using knot theory invariants?

The following questions deal with the possibility of classifying wild Cantor sets in \( \mathbb{R}^3 \) using various properties.

**Question 9.** Is there a way of classifying wild Cantor sets in \( \mathbb{R}^3 \) using local genus and other geometric properties?

**Question 10.** Can one use the volume of the hyperbolic 3-manifolds \( M^3 = S^3 \setminus X \) where \( X \) is a wild Cantor set to distinguish between classes of wild Cantor sets?

The following questions are about the relationship of Hausdorff dimension to various types of Cantor sets.

**Question 11.** Can two rigid Cantor sets have different Hausdorff dimensions? How does Hausdorff dimension detect rigidity of Cantor sets?

**Question 12.** Is there a rigid Cantor set of minimal Hausdorff dimension?

**Question 13.** Can two Cantor sets of different genus have the same Hausdorff dimension? How are Hausdorff dimension and genus of Cantor sets related?

The final few questions deal with homotopy groups of the complement of wild Cantor sets.

**Question 14.** Can two different (rigid) Cantor sets have the same fundamental groups of the complement?

**Question 15.** Which groups can occur as the fundamental groups of (rigid) wild Cantor set complements?

**References**


