QUANTUM LOGICS AS UNDERLYING STRUCTURES OF GENERALIZED PROBABILITY THEORY

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1 INTRODUCTION

As convincingly argued in [Mackey, 1963; Beltrametti and Cassinelli, 1981; Gudder, 1979; Varadarajan, 1985; Sarymsakov et al., 1983; Svozil, 1998], etc., one of the structures especially suited for considering a general (quantum) probability theory is that of a countably complete orthomodular poset (a "quantum logic", a term having its origin in the seminal paper by Birkhoff and von Neumann [Birkhoff and von Neumann, 1936]). From the point of view of theoretical physics, this structure seems to be well suited for dealing with the phenomenon of noncompatibility. From the purely mathematical point of view, it is challenging too because it permits a uniform generalization of probability theory on both projection lattices and Boolean algebras.

The conceptual generality inherent in the quantum logic approach opens a new field for interesting combinatorial and measure-theoretic considerations which, in turn, may enrich the study of quantum theory by which they were motivated. (In the short appendices we consider the uncertainty relations and the Bell inequalities as matters related to quantum stochastics.)

This chapter adds some novelties — both in the sense of updating and in style — to the stochastic results published in the book [Pták and Pulmannová, 1991]. Conceptually, the treatment presented here to a certain extent overlaps with the book cited, of course. Our intention was to make this presentation as self contained as possible. In this, however, we had to make a compromise between size and readability. The reader will see that there are some important technical results which we use in our considerations and for which we refer him to other sources. Certain important and interesting topics related to quantum probability theory such as e.g. quantum measurements [Busch et al., 1991] (see also [Pulmannová, 1994b]) or unsharp quantum logics [Dalla Chiara et al., 2004] are not included, and concerning the probability theory on projection lattices the reader is referred to [Hamhalter, 2003].
2 BASIC DEFINITIONS AND FACTS

In this section, we introduce basic definitions and results which we will need in the sequel. The emphasis is on the intrinsic properties of quantum logics and on the compatibility relation. More details can be found in [Beran, 1985; Beltrametti and Cassinelli, 1981; Kalmbach, 1983; Piron, 1976; Pták and Pulmannová, 1991; Rédei, 1998; Varadarajan, 1983].

DEFINITION 1. A partially ordered set \((L, \leq)\) endowed with a unary operation \(^{'}: L \rightarrow L\) is called a quantum logic, or simply a logic, if

(i) there exists a greatest element, 1, in \(L\);

(ii) \(a \leq b \implies a' \geq b'\ (a, b \in L)\);

(iii) \(a'' = a\ (a \in L)\);

(iv) \(a \lor a' = 1\ (a \in L)\);

(v) for every sequence \((a_i)_i\) of elements of \(L\) such that \(a_i \leq a'_j\) whenever \(i \neq j\) the supremum \(\bigvee_i a_i\) exists in \(L\);

(vi) if \(a \leq b\ (a, b \in L)\) then there is a \(c \in L\) with the properties \(c \leq a'\) and \(b = a \lor c\).

In the above definition, the symbols \(\lor\) and \(\land\) denote supremum and infimum in \(L\) respectively, if they exist. Property (vi) is called the orthomodular law. A quantum logic is alternatively called an orthomodular \(\sigma\)-poset. If property (v) holds only for finite sequences, \(L\) is called an orthomodular poset. Property (ii) implies the existence of a smallest element of \(L\), which we denote by 0. Obviously, \(0 = 1'\). It may be noted that (iv) follows from (vi) and, also, (v) for finite sequences follows from (vi), but economy in our axiomatic setup is not our aim. Property (ii) implies that the de Morgan laws hold in \(L\): if \(\forall a_i\) exists in \(L\), then \(\land a'_i\) exists in \(L\) and the equality \((\forall a_i)' = \land a'_i\) holds true.

DEFINITION 2. Two elements \(a, b \in L\) are said to be orthogonal (written \(a \perp b\)) if \(a \leq b'\).

DEFINITION 3. Two elements \(a, b \in L\) are said to be compatible (written \(a \leftrightarrow b\)) if there are pairwise orthogonal elements \(a_1, b_1, c\) in \(L\) such that \(a = a_1 \lor c\) and \(b = b_1 \lor c\).

LEMMA 4. If \(a \perp b\ (a, b \in L)\), then \((a \lor b) \land a' = b\).

Proof. If \(a \perp b\), then \((a \lor b) \land a'\) exists, in view of \(a \perp (a \lor b)'\) and \((a \lor (a \lor b)')' = a' \land (a \lor b)\). Clearly, \(b \leq (a \lor b) \land a'\). The orthomodularity implies that \((a \lor b) \land a' = b \lor d\), where \(d \perp b\). From \(d \leq (a \lor b) \land a' \leq a'\) we infer that \(d \leq a' \land b' = (a \lor b)'\), but this together with \(d \leq a \lor b\) yields \(d = 0\).

LEMMA 5. For any \(a, b \in L\),
(i) if \( a \perp b \), then \( a \leftrightarrow b \);

(ii) if \( a \leq b \), then \( a \leftrightarrow b \);

(iii) if \( a \leftrightarrow b \), then \( \{a, a', b, b'\} \) is a set of pairwise compatible elements.

**Proof.** It suffices to prove that \( a \leftrightarrow b \) implies \( a' \leftrightarrow b \). Let \( a \leftrightarrow b \), then \( a = a_1 \lor c \), \( b = b_1 \lor c \), where \( a_1, b_1, c \) are pairwise orthogonal. Then by Lemma 4, \((a \lor b_1) \land b_1' = a\), hence \( a' = (a' \land b_1') \lor b_1 \), and therefore the elements \( c, b_1, (a \lor b_1)' \) are pairwise orthogonal.

**LEMMA 6.** If \( a \leftrightarrow b \), then \( a \lor b \) and \( a \land b \) exist in \( L \) and \( a \lor b = a_1 \lor b_1 \lor c \), \( a \land b = a_1 \land b_1 \land c \), where \( a_1, b_1, c \) are the mutually orthogonal elements in the definition of compatibility. Moreover, \( a_1 = a \land b' \), \( b_1 = b \land a' \).

**Proof.** By orthogonality, \( a_1 \lor b_1 \lor c \) exists, and this element clearly dominates both \( a \) and \( b \). On the other hand, if \( e \in L \) dominates \( a \) and \( b \), then \( a_1 \lor c \leq e \), \( b_1 \lor c \leq e \) and this implies that \( a_1 \lor b_1 \lor c \leq e \). We see that \( a_1 \lor b_1 \lor c = a \lor b \).

The existence of \( a \land b \) follows from the fact that \( a \land b = (a' \lor b')' \) and from Lemma 5 (iii). By Lemma 4, \( a \land b = (a_1 \lor c) \land b \leq (b_1 \lor c) \land b = c \), which entails that \( a \land b = c \). Finally, \( a_1 \leq a \land b' \), hence \( a_1 \leq a \land b' \). On the other hand, \( a \land b' = (a_1 \lor c) \land b' \leq (a_1 \lor c) \land c' = a_1 \). The proof of \( b_1 = b \land a' \) is analogous.

**LEMMA 7.** For \( a, b \in L \), \( a \leftrightarrow b \) if and only if there is a \( c \in L \) with \( c \leq a, c \leq b \) and \( a \land c' \leq b' \).

**Proof.** Suppose \( a \leftrightarrow b \). Then \( a = a_1 \lor c, b = b_1 \lor c \), where \( c \leq a, c \leq b \) and \( a \land c' = a_1 \leq b' \).

Conversely, let \( c \leq a, c \leq b \) and let \( a \land c' \leq b' \). From the orthomodularity, there are \( a_1, b_1 \) such that \( a = a_1 \lor c, b = b_1 \lor c \), where \( a_1 \perp c, b_1 \perp c \). Moreover, \( a_1 = a \land c' \leq b' \leq b_1' \). It follows that \( a \leftrightarrow b \).

**LEMMA 8.** If \( b \leftrightarrow a_i, i = 1, 2, \ldots \) and the elements \( \bigvee_{i=1}^{\infty} a_i, \bigwedge_{i=1}^{\infty} a_i \land b \) exist in \( L \), then \( b \leftrightarrow \bigvee_{i=1}^{\infty} a_i \) and \( b \land \bigwedge_{i=1}^{\infty} a_i = \bigwedge_{i=1}^{\infty} a_i \land b \).

**Proof.** Obviously, \( \bigvee_{i=1}^{\infty} a_i \land b \) is not greater than \( \bigvee_{i=1}^{\infty} a_i \) and \( b \). By Lemma 7, \( b \leftrightarrow a_i \) implies that \( b \land (b \land a_i)' \leq a_i' \) for all \( i \). This entails that \( b \land (\bigvee_{i=1}^{\infty} b \land a_i)' \leq (b \land (b \land a_i)') \leq a_i' \) for all \( i \) and hence \( b \land (\bigvee_{i=1}^{\infty} b \land a_i)' \leq (\bigvee_{i=1}^{\infty} a_i)' \). This implies again by Lemma 7 that \( b \leftrightarrow \bigvee_{i=1}^{\infty} a_i \). The rest of the proof can be obtained by reasoning as in the second part of the proof of Lemma 7 with \( c = \bigwedge_{i=1}^{\infty} a_i \land b \) and an application of Lemma 6.

In what follows, let us denote by \([a, b]\) the set \( \{c \in L : a \leq c \leq b\} \), where \( a, b \in L \), \( a \leq b \).

**LEMMA 9.** Let \( a \in L \), \( a \neq 0 \). Then the set \([0, a]\) with the ordering inherited from \( L \) and with the orthocomplementation defined by setting \( b^\circ := b' \land a, b \in [0, a] \), is a logic.
Proof. Let \( b \leq a \). Then \( b' \land a \) exists by orthomodularity. The inequalities \( b_1 \leq b_2 \leq a \) imply \( b_2^a \leq b_1^a \), and thus \( b \lor b^a = b \lor (b' \land a) = a \) by orthomodularity, too. Also, \( (b^a)' = (b' \land a)' \land a = (b \lor a') \land a = b \) by Lemma 4. This proves that \( b \mapsto b^a \) is an orthocomplementation in \([0, a]\). Orthomodularity follows from orthomodularity in \( L \).

The logic of Lemma 9 will be denoted by \( L[0, a] \).

**Lemma 10.** If for every \( a, b \in L \) the supremum \( a \lor b \) exists in \( L \) then the logic \( L \) is a \( \sigma \)-lattice.

**Proof.** Let \( (a_i)_i \) be a sequence of elements of \( L \). Put \( b_1 = a_1, \ldots, b_n = a_1 \lor a_2 \lor \cdots \lor a_n (n \in U) \). Then \( b_1, b_2 \land b_1', \ldots \) is a sequence of orthogonal elements, hence its supremum, \( b \), exists. It is easily seen that \( b = \bigvee_{i=1}^{\infty} a_i \).

Note that a logic \( L \) becomes a Boolean \( \sigma \)-algebra if it is a distributive \( \sigma \)-lattice. The next theorem provides a characterization of Boolean \( \sigma \)-algebras among logics.

**Theorem 11.** A logic \( L \) is a Boolean \( \sigma \)-algebra if and only if \( a \leftrightarrow b \) for all \( a, b \in L \).

**Proof.** Necessity is clear. For sufficiency, if \( a \leftrightarrow b \) for all \( a, b \in L \), then by Lemma 6, \( a \lor b \) and \( a \land b \) exist in \( L \) for all \( a, b \in L \). By Lemma 10, \( L \) is a \( \sigma \)-lattice. Distributivity follows from Lemma 8.

A logic \( L \) which is also a lattice is called a **lattice logic**. Obviously, a lattice logic is an orthomodular \( \sigma \)-lattice.

Let us say that \( (a_1, a_2, a_3) \subset L \) is a **distribution triple** if \( a_i \land (a_j \lor a_k) = (a_i \land a_j) \lor (a_i \land a_k) \) for every permutation \( \{i, j, k\} \) of the set \( \{1, 2, 3\} \).

**Theorem 12** (Foulis–Holland). Let \( a, b, c \) be elements in a lattice logic \( L \) such that a chosen one is compatible with the other two. Then \( (a, b, c) \) is a distribution triple.

**Proof.** Assume, e.g., that \( a \leftrightarrow b, a \leftrightarrow c \). By Lemma 8 we have \( a \leftrightarrow b \lor c \) and \( a \land (b \lor c) = (a \land b) \lor (a \land c) \). It suffices to prove that \( (a \lor b) \land c = (a \land c) \lor (b \land c) \). By the De Morgan laws this is equivalent to \( (a \lor b) \lor c = (a \lor c) \lor (b \lor c) \). But \( (a \lor c) \lor (b \lor c) = (a \lor c) \lor [(a \land (b \lor c) \lor a' \land (b \lor c)] = a \land (b \lor c) \lor c = (a \lor b) \lor c \), where we used Lemma 8 and compatibility of \( a \lor c \) with \( a \land (b \lor c) \) and \( a' \land (b \lor c) \).

A subset \( M \) of a lattice logic \( L \) which has the property that among any three arbitrarily chosen elements of \( M \) there is one which is compatible with the other two, is called a **Foulis-Holland set**. Greechie [Greechie, 1979] (see also [Kalmbach, 1983] for a simpler proof) proved that the sublattice of \( L \) generated by a Foulis-Holland set \( M \) is distributive.

The following lemma is a strengthening of Lemma 8 for lattice logics.
THEOREM 13. Let $L$ be a lattice logic. Let $M$ be a subset of $L$ such that $\bigvee M := \{m : m \in M\}$ exists in $L$, and let $b \leftrightarrow m$ for all $m \in M$. Then $b \leftrightarrow \bigvee M$ and $\bigvee\{b \land m : m \in m\} = (\bigvee M) \land b$.

**Proof.** Evidently, $(\bigvee M) \land b \geq m \land b$ for all $m \in M$. Let $u \in L$ be such that $u \geq m \land b$ for all $m \in M$. Then $z := u \land (\bigvee M) \land b \geq m \land b$ for all $m \in M$. We will show that $z = (\bigvee M) \land b$. From $b \leftrightarrow m$ for all $m \in M$ it follows that $b \land (b') \land (\bigvee M)' \leq b \land (b' \land (\bigvee M)') = b \land m' \leq b \land (\bigvee M)'$. By Lemma 7, $b \leftrightarrow \bigvee M$. From orthomodularity we have that $(\bigvee M) \land b = z \land d$, where $d \perp z$. Thus $d \leq (\bigvee M) \land b$ and $d \leq z' \leq m' \land b'$. This entails that $d \leq (m' \land b') \land b = m' \land b$ for all $m \in M$. Hence $d \leq (\bigvee M)'$, which together with $d \leq \bigvee M$ entails that $d = 0$. We have shown that $u \geq (\bigvee M) \land b$, so that $(\bigvee M) \land b$ is the supremum of the set $\{b \land m : m \in M\}$. \hfill \ensuremath{\Box}

**DEFINITION 14.** Let $L_1$ and $L_2$ be logics. A mapping $h : L_1 \to L_2$ is said to be a $\sigma$-homomorphism if

(i) $h(1) = 1$;
(ii) $h(a') = h(a)'$ ($a \in L_1$);
(iii) $h(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} h(a_i)$ for every sequence $(a_i)$ of mutually orthogonal elements of $L_1$.

A bijective $\sigma$-homomorphism $h$ such that $h^{-1}$ is also a $\sigma$-homomorphism is called an isomorphism. If $L_1 = L_2$, then an isomorphism is called an automorphism.

A $\sigma$-homomorphism is a lattice $\sigma$-homomorphism if $h(a \land b) = h(a) \land h(b)$ whenever $a \land b$ exists in $L_1$.

If the condition (iii) is required to be satisfied only for finite orthogonal sequences, then $h$ is called a homomorphism.

Let us observe that if $h : L_1 \to L_2$ is a $\sigma$-homomorphism and $(a_i)_i$ is an arbitrary sequence of elements in a lattice logic $L_2$, then $h(\bigvee_i a_i) = \bigvee_i h(a_i)$. Note also that if $L_1$ is a lattice logic and $L_2$ is an arbitrary logic, then every $\sigma$-homomorphism $h : L_1 \to L_2$ is a lattice $\sigma$-homomorphism exactly when $L_1$ is a Boolean $\sigma$-algebra [Mañasová, 1981].

**DEFINITION 15.** A subset $L_1$ of a logic $L$ is called a sublogic when $L_1$ is a logic with the ordering and complementation inherited from $L$ and when the identity mapping $id : L_1 \to L$ is an injective $\sigma$-homomorphism. A sublogic is called a lattice sublogic if the following implication holds: Whenever $(a_i)_i$ is a sequence of elements in $L_1$ such that $\bigvee_i a_i$ exists in $L_1$, then $\bigvee_i a_i$ belongs to $L_1$.

A sublogic $L_1$ of a logic $L$ is called a Boolean sublogic if (i) for every sequence $(a_i)_i$ of $L_1$ the supremum $\bigvee_i a_i$ exists in $L$ and belongs to $L_1$; (ii) every $(a, b, c) \subset L_1$ is a distributive triple.

**DEFINITION 16.** We say that a logic $L$ is regular if for every $a, b, c$ in $L$ which are pairwise compatible we have $a \leftrightarrow b \land c$ (equivalently, $a \leftrightarrow b \lor c$).
Let $A$ be an arbitrary subset of a logic $L$. Put $$C(A) := \{a \in L : a \leftrightarrow b \text{ for every } b \in A\}.$$ We easily see that, for every $A, B \subseteq L$,

(i) $A \subseteq B \implies C(B) \subseteq C(A)$;

(ii) $A \subseteq C(C(A))$;

(iii) $CC(C(A)) = C(CC(A)) = C(A)$.

The set $C(L)$ is called the center of $L$.

**Lemma 17.** For every subset $A \subseteq L$, the set $C(A)$ is a sublogic of $L$.

**Proof.** Evidently, 0 and 1 belong to $C(A)$. If $a \in C(A)$, then also $a' \in C(A)$ by Lemma 5. If $(a_i)_{i \in \mathbb{N}}$ is a sequence of pairwise orthogonal elements in $C(A)$, then $\bigvee_{i \in \mathbb{N}} a_i$ exists in $L$, and for every $b \in A$ we have $b \leftrightarrow a_i, \ i = 1, 2, \ldots$. Hence $b \land a_i$ exists. As $(b \land a_i)_{i \in \mathbb{N}}$ is an orthogonal sequence, $\bigvee_{i \in \mathbb{N}} b \land a_i$ exists in $L$. By Lemma 8, $b \leftrightarrow \bigvee_{i \in \mathbb{N}} a_i$, and hence $\bigvee_{i \in \mathbb{N}} a_i \in C(A)$.

**Lemma 18.** If $B$ is a subset of pairwise orthogonal elements in a regular logic $L$, then $C(C(B))$ is a Boolean sublogic of $L$.

**Proof.** By Lemma 17, $C(C(B))$ is a sublogic of $L$. Since $B$ is a pairwise compatible set, we have $B \subseteq C(B)$, which entails that $C(C(B)) = C(C(C(B)))$, so that $C(C(B))$ is a compatible set. Let $a, b \in C(C(B))$ and $c \in C(B)$. Then $a, b, c$ are pairwise compatible. Then regularity implies that $c \leftrightarrow a \lor b$, so that $a \lor b \in C(C(B))$. It follows that $C(C(B))$ is a lattice. Moreover, $(a, b, c) \in C(C(B))$ is a distributive triple for every $a, b, c$.

**Proposition 19.** The center $C(L)$ of any logic $L$ is a Boolean sublogic of $L$.

**Proof.** By Lemma 17, $C(L)$ is a sublogic of $L$. If $a, b \in C(L)$, then $a \leftrightarrow b$, therefore $a \lor b$ exists in $L$. Moreover, $a \land c$ and $b \land c$ exist in $L$ for all $c \in L$, and in addition, $a \land c \leftrightarrow b \land c$. By Lemma 8, $a \lor b \in C(L)$. Since $a \leftrightarrow b$ for all $a, b \in C(L)$, $C(L)$ is a Boolean sublogic.

**Theorem 20.** Let $L$ be a regular logic. Let $K := \{a_\alpha : \alpha \in I\}$ be a subset of pairwise compatible elements in $L$. Then there exists a Boolean sublogic $B$ of $L$ such that $K \subseteq B \subseteq L$.

**Proof.** We have $K \subset C(C(K))$, and by Lemma 18, $C(C(K))$ is a Boolean sublogic of $L$.

**Definition 21.** A logic $L$ is called separable if every set of nonzero pairwise orthogonal elements in $L$ is at most countable.

**Proposition 22.** Let $L$ be a separable lattice logic. For every set $A \subseteq L$ there exists a countable subset $(a_i)_{i \in \mathbb{N}}$ such that $\bigvee A = \bigvee_{i=1}^\infty a_i$. Consequently, $L$ is a complete lattice.
Proof. Let \( A \subseteq L \). Let \( B \) denote the set of all suprema of at most countable subsets of \( A \). Let \( C := \{ b \land c' : b \in B, c \leq b \} \). By Zorn's lemma, there exists a maximal subset of pairwise orthogonal elements in \( C \), let us say \( \{ b_1 \land c_1', b_2 \land c_2', \ldots \} \), which is at most countable. Let \( b = \bigvee_{i=1}^{\infty} b_i \). Then clearly \( b \in B \). If there is \( a \in A \) such that \( a \nleq b \), then \( 0 \neq (b \lor a) \land b' \in C \). Then the inequalities \( b_i \land c_i' \leq b \leq (b \lor a) \lor b \) for \( i = 1, 2, \ldots \) imply that \( (b \lor a) \land b' \perp b_i \land c_i', i = 1, 2, \ldots , \) which contradicts the maximality of the set \( \{ b_i \land c_i' \}_{i=1}^{\infty} \). This proves that \( b = \bigvee A \).

DEFINITION 23. An element \( a \) in a logic \( L \) is called an atom if \( a \neq 0 \) and the inequality \( b \leq a \) implies either \( b = 0 \) or \( b = a \).

DEFINITION 24. A logic \( L \) is called atomic if for every \( b \in L \), \( b \neq 0 \), there is an atom \( a \in L \) with \( a \leq b \). A logic \( L \) is called atomistic if every element in \( L \) can be written as a supremum of all atoms that are less than or equal to \( L \).

PROPOSITION 25. A lattice logic is atomic if and only if it is atomistic.

Proof. Let \( L \) be an atomic lattice logic. Let \( b \in L \), \( b \neq 0 \) and let \( A \) be the set of all atoms lying under \( b \). Then \( A \neq \emptyset \). Let \( c \in L \) be such that \( a \leq c \) for all \( a \in A \). Then \( c \leq b \land c \), and by orthomodularity, \( b = (b \land c) \lor (b \land (b' \lor c')) \). If \( b \land (b' \lor c') \neq 0 \), then, by the atomicity of \( L \), there is an atom \( e \leq b \land (b' \lor c') \). But \( e \leq b \) implies \( e \leq b \land c \), in contradiction with \( e \leq b' \lor c' \). Hence \( b = b \land c \), and therefore \( b = \bigvee A \).

Let us note in concluding this paragraph that Prop.25 does not have to hold for general logics (see [Greechie, 1969]).

3 COMPATIBLE SUBSETS OF A LOGIC

In the quantum logic approach to quantum mechanics, a quantum logic \( L \) is interpreted as the set of all experimentally verifiable propositions about a physical system. The compatible subsets of \( L \) should correspond to those propositions that can be simultaneously verified, and hence can be dealt with as classical propositions. Generally, we may assume that the elements of a subset \( M \) of \( L \) are compatible if they can be embedded into a Boolean sublogic of \( L \), in accord with the assumption that the logic of a classical system is a Boolean \( \sigma \)-algebra. In this section, we will find intrinsic characterizations of compatible subsets of a logic.

If a logic \( L \) is regular, then by Theorem 20, every pairwise compatible subset of elements of \( L \) is contained in a Boolean sublogic of \( L \). The following example shows that this need not hold in an arbitrary logic.

EXAMPLE 26. Let \( M = \{1, 2, 3, 4, 5, 6, 7, 8\} \) and let \( L \) be the set of all subsets of \( M \) with an even number of elements. Then \( L \) ordered by inclusion and endowed with set-theoretical orthocomplementation is a logic. Let \( A = \{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{2, 3, 5, 7\} \). It is easy to check that elements of \( A \) are
pairwise compatible. But they cannot be contained in a Boolean sublogic of \( L \) since \( \{1, 2, 3, 4, 5, 6\} \wedge \{2, 3, 5, 7\} \) does not exist in \( L \).

As a result, a stronger condition of compatibility is needed to characterize those subsets of a logic that can be embedded into Boolean sublogics of \( L \). Such a condition, so-called strong compatibility, was found independently in [Guz, 1971] and [Neubrunn, 1974]. Let us introduce it in the next two definitions.

**DEFINITION 27.** Let \( M \) be a subset of a logic \( L \). We say that \( a, b \in M \) are **compatible in \( M \)**, written \( a \overset{M}{\leftrightarrow} b \), if there are pairwise orthogonal elements \( a_1, b_1, c \) in \( M \) such that \( a = a_1 \vee c \) and \( b = b_1 \vee c \).

**DEFINITION 28.** A subset \( A \) of a logic \( L \) is said to be **strongly compatible** if for every two elements \( a, b \in A \), \( a \overset{L(A)}{\leftrightarrow} b \), where \( L(A) \) is the smallest sublogic of \( L \) that contains \( A \).

Obviously, the smallest sublogic of \( L \) that contains \( A \), \( L(A) \), exists (\( L(A) \) is the intersection of all sublogics of \( L \) that contain \( A \)). Recall that the smallest sublogic of \( L \) that contains \( A \) is called the sublogic of \( L \) generated by \( A \).

**THEOREM 29.** Let \( A \subseteq L \), and let \( L(A) \) be the sublogic of \( L \) generated by \( A \). Then \( L(A) \) is a Boolean sublogic of \( L \) if and only if \( A \) is strongly compatible.

**Proof.** If \( L(A) \) is a Boolean sublogic, then \( A \subseteq L(A) \) implies that for every \( a, b \in A \), \( a = a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b') \) by distributivity in \( L(A) \). Since \( L(A) \) is a Boolean sublogic, \( a \wedge b, a \wedge b' \in L(A) \), hence \( a \overset{L(A)}{\leftrightarrow} b \).

Conversely, assume that \( A \) is strongly compatible. For \( a \in A \), put \( B_a := \{ b \in L(A) : a \overset{L(A)}{\leftrightarrow} b \} \). By the properties of compatibility, \( B_a \) is a sublogic of \( L \), and since \( A \subseteq B_a \) by strong compatibility, we have \( L(A) \subseteq B_a \) for every \( a \in A \). Now let \( b \in L(A) \), and put \( B_b := \{ c \in L(A) : b \overset{L(A)}{\leftrightarrow} c \} \). The first part of this proof implies that \( A \subseteq B_b \), and hence \( L(A) \subseteq B_b \) for all \( b \in L(A) \). This entails that every pair \( a, b \in L(A) \) is compatible in \( L(A) \). As a consequence, \( L(A) \) is a Boolean sublogic of \( L \).

Obviously, strong compatibility implies pairwise compatibility. The next example shows that strong compatibility is not equivalent to pairwise compatibility, and also that strong compatibility is not necessary for a subset of \( L \) to be embeddable into a Boolean sub logic of \( L \).

**EXAMPLE 30.** Let \( M = \{1, 2, 3, 4\} \) and let \( L \) be the Boolean algebra of all subsets of \( M \). Let \( A = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\} \). The smallest sublogic \( L(A) \) containing \( A \) consists of all subsets with an even number of elements. The elements of \( A \) are pairwise compatible in \( L \) but not strongly compatible, (e.g., \( \{1, 2\} \cap \{2, 3\} = \{2\} \) which does not belong to \( L(A) \)).

In what follows, we will use the following notation. Let \( D = \{-1, 1\} \), and let \( A \subseteq L \). Write \( A' := \{ a' : a \in M \} \) and let \( a^1 := a, a^{-1} := a' \). Put \( D := \{-1, 1\} \).
Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $L$ such that for every $d \in D^n$ the infimum $a_1^{d_1} \wedge a_2^{d_2} \wedge \cdots \wedge a_n^{d_n}$ exists in $L$. Then put
\[
F_A := \{a_1^{d_1} \wedge a_2^{d_2} \wedge \cdots \wedge a_n^{d_n} : d \in D\}
\]
and set
\[
com(a_1, a_2, \ldots, a_n) = \bigvee\{f : f \in F_A\}.
\]

**DEFINITION 31.** Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of a logic $L$. We say that $A$ is a **compatible set** if all elements in $F_A$ exist and $com(a_1, a_2, \ldots, a_n) = 1$. If $A$ is an arbitrary subset of $L$ then $A$ is said to be compatible if all finite subsets of $A$ are compatible. The element $com(a_1, a_2, \ldots, a_n)$ (if it exists) is called the **commutator** of the set $\{a_1, a_2, \ldots, a_n\}$.

If $L$ is a lattice, then $com(A)$ exists for any finite subset $A \subset L$. For any two elements we have $com(a, b) = (a \wedge b) \vee (a' \wedge b') \vee (a \wedge b')$. This commutator in orthomodular lattices was introduced in [Marsden, 1970] (see also [Chevalier, 1984; Chevalier, 1989; Matoušek, 1992], etc.).

**THEOREM 32.** A subset $A$ of $L$ is contained in a **Boolean sublogic** of $L$ if and only if $A$ is a compatible set.

**Proof.** If there is a Boolean sublogic $L_0$ of $L$ such that $A \subset L_0$, then $A$ is a compatible set.

Assume that $A$ is a compatible subset of $L$. If $M = \{a_1, a_2, \ldots, a_n\}$ is a finite subset of $A$, then $F_M = \{a_1^{d_1} \wedge a_2^{d_2} \wedge \cdots \wedge a_n^{d_n} : d \in D\}$ is a set of pairwise orthogonal elements of $L$ the supremum of which equals $1$. Let $B(M)$ denote the set of suprema of all subsets of $F_M$. It is easily verified that $B(M)$ is a Boolean sublogic of $L$. Put $B = \bigcup\{B(M) : M \text{ a finite subset of } A\}$. Then $B$ is also a Boolean sublogic of $L$. Indeed, if $a_1, a_2, a_3 \in B$ and if $M_1, M_2, M_3$ are finite subsets of $A$ with $a_i \in B(M_i), i = 1, 2, 3$, then $M_1 \cup M_2 \cup M_3$ is a finite subset of $A$, and $B(M_1 \cup M_2 \cup M_3)$ contains all $a_1, a_2, a_3$. It follows that any three elements of $B$ are contained in a Boolean sublogic of $L$. Therefore $B$ is a Boolean subalgebra of $L$. We still need to verify that $B$ is closed under the formation of suprema of countable subsets. Let $L_0$ be the sublogic of $L$ generated by $B$. We will show that $L_0$ is a Boolean sublogic of $L$. Take $a \in B$ and put $K_a := \{b \in L_0 : b \leq a\}$. By Lemma 17, $K_a$ is a sublogic of $L_0$. Since $B$ is Boolean, we have $B \subset K_a$. Thus $K_a = L_0$ for all $a \in B$. Further, take $b \in L_0$ and put $K_b := \{c \in L_0 : c \leq b\}$. Then, again, $K_b$ is a sublogic of $L_0$. By the first part of this proof, $B \subset K_b$. It follows that $K_b = L_0$ for all $b \in L_0$. Hence we have $a \leftrightarrow b$ in $L_0$ for all $a, b \in L_0$. By Theorem 11, $L_0$ is Boolean $\sigma$-algebra, hence $L_0$ is a Boolean sublogic of $L$.

The following condition was introduced in [Brabec, 1979].

**DEFINITION 33.** A subset $A$ of a logic $L$ is $f$-compatible if for every finite subset $F = \{a_1, a_2, \ldots, a_n\}$ of $A$ there exists a finite set $G = \{g_1, g_2, \ldots, g_k\}$ of pairwise orthogonal elements of $L$ such that for every $i = 1, 2, \ldots, n$ there is a subset $H_i \subset G$ such that $a_i = \bigvee\{f_j : f_j \in H_i\}$. The set $G$ is called an orthogonal cover of $F$. 

The following statement can be proved by a technique similar to the proof of Theorem 32 ([Pták and Pulmannová, 1991, Prop. 1.3.22], see also [Brabec, 1979; Brabec and Pták, 1982]).

**PROPOSITION 34.** Let \( A \) be a finite subset of \( L \). Then \( A \) is compatible if and only if it is \( f \)-compatible.

By Lemma 9, for any \( b \in L \), the interval \([0, b]\) can be viewed as a logic with the smallest element, \( 0 \), the greatest element, \( b \), and with the relative orthocomplementation \( x^b = x' \land b \). In the next lemma, we prove that compatibility in \( L \) and compatibility in \([0, b]\) are equivalent.

**LEMMA 35.** Let \( b \in L \). The elements \( a_1, a_2, \ldots, a_n \) in the interval \([0, b]\) are compatible in \( L \) if and only if they are compatible in \([0, b]\).

**Proof.** If \( \text{com}(a_1, a_2, \ldots, a_n) \) exists and equals to 1, the it is easy to see that \( \text{com}(a_1, a_2, \ldots, a_n) \land b \) is the commutator of \( a_1, a_2, \ldots, a_n \) taken in \([0, b]\), and since it is equal to \( b \), the compatibility in \([0, b]\) follows.

Conversely, if elements are compatible in \([0, b]\), then they have an orthogonal cover, which means that they are \( f \)-compatible and hence by Proposition 34 they are compatible in \( L \).

It is easily seen that for a two-element set \( A = \{a_1, a_2\} \) the notions of pairwise compatibility, \( f \)-compatibility and compatibility coincide. We have already shown that pairwise compatibility is strictly weaker than compatibility. The next theorem (see [Pták and Pulmannová, 1991, Theorem 1.3.25]) shows that also "triplewise", etc. compatibility is too weak to guarantee compatibility.

**THEOREM 36.** Let \( n (n \geq 2) \) be a natural number. Then there exists a (finite) logic \( L_n \) with a subset \( A_n \) such that the following statements hold:

(i) the set \( A_n \) is not compatible in \( L_n \);

(ii) every subset of \( A_n \) which contains less that \( n \) elements is compatible in \( L_n \).

Let us recall ([Harding, 1998; Harding, 1839–1862; Pták and Pulmannová, 1991], etc.) that a logic \( L \) is said to be regular if for any set \( \{a, b, c\} \subseteq L \) of pairwise compatible elements we have \( a \leftrightarrow b \lor c \). As seen before, there are logics that fail to be regular (Example 26). Interestingly, projection logics are usually regular (see e.g. [Flachsmeyer and Katrnoška, 1979; Harding, 1998] and [Pták and Weber, 2001]).

**PROPOSITION 37.** Every lattice logic is regular.

**Proof.** The proof follows immediately from Lemma 8.

**PROPOSITION 38.** A logic \( L \) is regular if and only if every pairwise compatible subset of \( L \) admits an enlargement to a Boolean sublogic of \( L \).
Proof. By Theorem 20, every pairwise compatible subset of a regular logic is contained in a Boolean sublogic. To prove the converse, assume that every pairwise compatible subset of a logic $L$ is compatible. Then if $a, b, c$ are pairwise compatible elements of $L$, then they are contained in a Boolean sublogic $L_0$ of $L$, hence $b \lor c$ exist in $L$ and belongs to $L_0$, and $a \leftrightarrow b \lor c$ in $L_0$ implies $a \leftrightarrow b \lor c$ in $L$.

PROPOSITION 39. (i) Let $A$ be a compatible subset of $L$. Among the Boolean sublogics of $L$ containing $A$ there is a maximal and a minimal one. (ii) The union of all maximal Boolean sublogics of $L$ is $L$. (iii) The intersection of all maximal Boolean sublogics of $L$ is $C(L)$.

Proof. Let $\mathcal{K}_A$ denote the collection of all Boolean sublogics of $L$ that contain $A$. By Theorem 32, $\mathcal{K}_A$ is nonempty. The intersection, $L_{\text{min}}$, of $\mathcal{K}_A$ is the smallest Boolean sublogic that contains $A$. Let us consider the collection $\mathcal{K}_A$ ordered by inclusion. By a standard application of Zorn's lemma the collection $\mathcal{K}_A$ contains a maximal element, $L_{\text{max}}$. This completes the proof of the statement. The statements (ii) and (iii) are obvious consequences of the statement (i).

DEFINITION 40. A maximal Boolean sublogic of $L$ is called a block of $L$.

In view of Proposition 39 every logic can be viewed as a union of its blocks. A more detailed analysis of the configuration of blocks in a logic can be found in [Greechie, 1971; Matouavsek, 2004] and [Navara and Rogalewicz, 1991].

4 STATES ON A LOGIC

DEFINITION 41. A state on a logic $L$ is a mapping $s : L \rightarrow [0, 1]$, where $[0, 1]$ denotes the unit interval of the real line $\mathbb{R}$, which satisfies the following conditions:

(i) $s(1) = 1$;

(ii) if $(a_i)_i$ is a sequence of pairwise orthogonal elements in $L$, then $s(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} s(a_i)$.

If $s_1$ and $s_2$ are states on $L$, then their convex combination $s = \alpha s_1 + (1-\alpha)s_2$, where $0 \leq \alpha \leq 1$, are also states on $L$. A state $s$ on $L$ is called pure if $s = \alpha s_1 + (1-\alpha)s_2$, $0 < \alpha < 1$ implies that $s = s_1 = s_2$. A state which is not pure is called a mixture. If for every $a \in L$, $s(a) \in \{0, 1\}$, we say that $s$ is a two-valued state.

Let $S(L)$ denote the set of all states on $L$ and let $S_2(L)$ denote the set of all two-valued states on $L$. Let us note that every $s \in S_2(L)$ is pure. Indeed, if we can write $s = \alpha s_1 + (1-\alpha)s_2$ for $\alpha \in (0, 1)$ and for distinct $s_1, s_2 \in S(L)$, then there is $0 < \alpha < 1$ with $s_1(a) \neq s_2(a)$, and the value $\alpha s_1(a) + (1-\alpha)s_2(a)$ is neither 0 nor 1. Moreover, if $L$ is a Boolean logic, then $s \in L_2(L)$ iff $s$ is pure. Indeed, if $s \in S(L)$ is such that $0 < s(b) < 1$ for some $b \in L$, we can put $s = \alpha s_1 + (1-\alpha)s_2$, where
\(\alpha = s(b)\) and \(s_1, s_2\) are states on \(L\) defined by the formulas \(s_1(a) = s(b)^{-1} s(a \wedge b)\) and \(s_2(a) = (1 - s(b))^{-1} s(a \wedge b')\). Thus \(s\) is not pure.

**Lemma 42.** The set \(S(L)\) is a \(\sigma\)-convex set, that is, if \(s_1, s_2, \ldots\), are states and \(c_i \geq 0\), \(\sum_{i=1}^{\infty} c_i = 1\), then \(s = \sum_{i=1}^{\infty} c_i s_i\) is also a state on \(L\).

**Proof.** It follows from the fact that \(s(a) = \sum_{i=1}^{\infty} c_i s_i(a) \leq \sum_{i=1}^{\infty} c_i = 1\) for every \(a \in L\).

**Definition 43.** We say that a state \(s\) on a logic \(L\) has the \Jauch-Piron property, or is a \Jauch-Piron state if the following implication holds: If \(s(a) = 1\) and \(s(b) = 1\) for some \(a, b \in L\), then there is \(c \in L\), \(c \leq a, c \leq b\) such that \(s(c) = 1\).

Equivalently, a state \(s\) is \Jauch-Piron if \(s(a) = 0 = s(b)\) implies that there is \(d \in L\), \(a, b \leq d\), and \(s(d) = 0\). If \(L\) is a lattice logic, then a state \(s\) is \Jauch-Piron iff \(s(a) = 1 = s(b)\) implies \(s(a \wedge b) = 1\) \((a, b \in L)\). In the study of \Jauch-Piron states we encounter new phenomena in (non-commutative) measure theory see e.g. [De Lucia and Pták, 1992; Bunce et al., 1985; Müller, 1993; Rüttimann, 1977; Pták, 1998].

**Definition 44.** We say that a set \(S\) of states on \(L\) is

1. **order determining** if \(s(a) \leq s(b)\) for all \(s \in S\) implies \(a \leq b\);
2. **unital** if for every \(a \in L\), \(a \neq 0\) there is \(s \in S\) such that \(s(a) = 1\);
3. **rich** if \(\{s \in S : s(a) = 1\} \subset \{s \in S : s(b) = 1\}\) implies \(a \leq b\). (Equivalently, if \(a \nleq b\) implies \(\exists s \in S : s(a) = 1, s(b) \neq 0\)).

We say that a logic \(L\) is

1. **unital** if it has a unital set of states,
2. **rich** if it has a rich set of states,
3. **rich\(^2\)** if it admits a rich set of \(\{0, 1\}\)-states.

**Lemma 45.** (i) A rich set \(S\) is unital and order determining. (ii) If \(L\) is a lattice logic, then a unital set \(M\) of \Jauch-Piron states is rich.

**Proof.** Part (i) is straightforward. For (ii) Let \(a \nleq b\). Then \(a = (a \wedge b) \vee c\) with \(c \neq 0, c \bot a \wedge b\). Let \(s(a) = 1 \implies s(b) = 1\). Then the \Jauch-Piron property implies that \(s(a) = 1 \implies s(a \wedge b) = 1\). By unitality there is \(s \in M\) such that \(s(c) = 1\). But then \(s(c) = 1 \implies s(a) = 1 \implies s(a \wedge b) = 1\), contradicting \(c \bot a \wedge b\).

As a consequence of orthomodularity we have \(a \leq b \implies s(a) \leq s(b)\). In the definition of a state, orthomodularity is not explicitly used. Consequently, the notion of a state makes sense for any partially ordered orthocomplemented set which
is often forced (see also [Dvurečenskij, 1993] and [Hamhalter and Pták, 1987]).

PROPOSITION 46. Let \((L, \leq,')\) be a partially ordered orthocomplemented set with the properties (i)-(v) of Definition 1. Let \(S\) be an order determining set of states on \(L\). Then \(L\) also has property (vi) of Definition 1, that is, \(L\) is a logic.

**Proof.** Let \(a \leq b, a, b \in L\). Then \(a \lor b'\) exists in \(L\), and \(s(a \lor b') = s(a) + s(b') = s(a) + 1 - s(b)\). For any \(s \in S\) we have

\[
s(a \lor (a \lor b')) = s(a) + s((a \lor b')') = s(a) + 1 - s(a \lor b') = s(b),
\]

and since \(S\) is order determining, this implies that \(a \lor (a \lor b') = b\). This is equivalent to (vi) of Definition 1.

The following definition introduces an important class of set-representable logics. It should be noted that even finite logics are not necessarily set-representable, see [Katrnoška, 1982; Ovchinnikov, 1997; Tkadlec, 1993; Sherstnev, 1968], etc.

**DEFINITION 47.** Let \(\Omega\) be a nonempty set and let \(\Delta\) be a collection of subsets of \(\Omega\). Then \(\Delta\) is said to be a **concrete logic** if the following conditions are satisfied:

1. \(\emptyset \in \Delta\);
2. if \(A \in \Delta\) then \(\Omega \setminus A \in \Delta\);
3. if \(\{A_i : i \in \mathbb{N}\} \subset \Delta\) is a countable family of pairwise disjoint subsets of \(\Omega\), then \(\bigcup \{A_i : i \in \mathbb{N}\} \in \Delta\)

It is interesting to note that the notion of a concrete logic has long been known in classical probability theory in connection with generating Borel sets (see e.g. [Olejček, 1995] for more details). There, it is called a Dynkin system of sets. Note also that there are Boolean logics (Boolean \(\sigma\)-algebras) which are not concrete [Pták and Pulmannová, 1991].

The following statement was proved in [Gudder, 1979].

**THEOREM 48.** A logic \(L\) is isomorphic (as a logic) to a concrete logic if and only if \(L\) is rich\(^2\).

**Proof.** Suppose first that \(L\) is isomorphic to a concrete logic \((\Omega, \Delta)\). Then we may identify \(L\) with \(\Delta\). Take \(A, B \in \Delta\) such that \(A \not\subseteq B\). Then \(A \setminus B \neq \emptyset\) and therefore we can choose a point \(p \in A \setminus B\). Consider a state \(s_p\) concentrated at \(p\) (i.e., \(s_p(C) = 1\) iff \(p \in C\)). Then \(s_p(A) = 1\) and \(s_p(B) = 0\). Thus, \(L\) is rich\(^2\).

Conversely, suppose that \(L\) is rich\(^2\). Put \(\Omega = S_2(L)\), and \(\Delta = \{A \in S_2(L) : \text{there is an } a \in L \text{ such that } A = \{s \in S_2(L) : s(a) = 1\}\}\). We show that \((\Omega, \Delta)\) is a concrete logic. If \(A = \{s \in S_2(L) : s(a) = 1\}\) then \(\Omega \setminus A = \{s \in S_2(L) : s(a') = 1\}\). Hence \(A \in \Delta\) implies \(\Omega \setminus A \in \Delta\). Let \(A_i = \{s \in S_2(L) : s(a_i) = 1\}, i = 1, 2, \ldots\),
and let $A_i$ be mutually disjoint sets ($i \in \mathbb{N}$). As $L$ is rich\(^2\), it follows that the elements $a_i, i \in \mathbb{N}$ are mutually orthogonal. We may then write $\bigcup_{i \in \mathbb{N}} A_i = \{s \in S_2(L) : s(\bigvee_{i \in \mathbb{N}} a_i) = 1\}$. Hence $(\Omega, \Delta)$ is a concrete logic.

The following proposition gives a characterization of compatibility in concrete logics.

**Proposition 49.** Let $\Delta$ be a concrete logic of subsets of $\Omega$. Then $A, B \in \Delta$ are compatible if and only if $A \cap B \in \Delta$.

**Proof.** Assume that $A \leftrightarrow B$. Then there are mutually disjoint sets $A_1, B_1, C$ in $\Delta$ such that $A = A_1 \cup C$, $B = B_1 \cup C$. Then $A \cap B = C \in \Delta$. Now assume that $A \cap B \in \Delta$. Then we have $A = (A \cap B) \cup (A \setminus (A \cap B))$, $B = (A \cap B) \cup (B \setminus (A \cap B))$, where the sets $A \cap B, A \setminus (A \cap B), B \setminus (A \cap B)$ are mutually disjoint sets. Since for disjoint elements of $\Delta$ the set-theoretical union $\cup$ coincides with the supremum in $\Delta$. It follows that $A \leftrightarrow B$.

The state space $S(L)$ of a logic can be rather poor (even for finite $L$ we can have $S(L) = \emptyset$—see [Greechie, 1971; Navara, 1994] and [Weber, 1994]) or rather bizarre ([Navara et al., 1988; Navara and Rogalewicz, 1988]). To illustrate that, let us consider the following four classes of (unital) logics, denoted by $L_1, L_2, L_3, L_4$:

- $L \in L_1$ if $a \not\not b$, there is a state $s \in S(L)$ such that $s(a) = 1$ and $s(b) = 1$;
- $L \in L_2$ if $a \not\not b$, and if we are given an $\varepsilon > 0$, then there is a state $s \in S(L)$ such that $s(a) = 1$ and $s(b) \geq 1 - \varepsilon$;
- $L \in L_3$ if $a \not\not b$, then there is a state $s \in S(L)$ such that $s(a) = 1 = s(b)$;
- $L \in L_4$ if $a \not\not b$, then there is a two-valued state $s \in S_2(L)$ such that $s(a) = 1 = s(b)$ and, moreover, the set $S_2(L)$ is unital for $L$.

**Theorem 50.** [Pták and Pulmannová, 1991, Th. 2.4.12] (i) $L \in L_1 \iff L$ is a rich logic. (ii) $L \in L_4 \iff L$ is a concrete logic. (iii) The inclusions $L_4 \subseteq L_3 \subseteq L_2 \subseteq L_1$ hold.

**Proof.** (i) We rephrase the definition of a rich logic: a logic $L$ is rich if for any $a, b \in L$ with $a \not\not b$ there is a state $s \in S(L)$ such that $s(a) = 1$ and $s(b) = 1$. Thus every rich logic obviously belongs to $L_1$. Now let $L \in L_1$ and let $a, b \in L$, $a \not\not b$. If $a \leftrightarrow b$, we have $a = a_1 \vee c, b = b_1 \vee c$, where $a_1, b_1, c$ are mutually orthogonal. Obviously, $a_1 \neq 0$. Take a state $s \in S(L)$ with $s(a_1) = 1$. Then we have $s(a) \geq s(a_1) = 1$. From $b \leq a_1$ we have $s(a'_1) = 0 \geq s(b)$. Hence $s(a) = 1$ and $s(b) = 0$.

(ii) Follows by (i) and Theorem 48.

(iii) The inclusions $L_4 \subseteq L_3 \subseteq L_2$ follow immediately from the definitions. It remains to verify that $L_2 \subset L_1$. Suppose that $L \in L_2$ and take elements $a, b \in L$. If $a \not\not b$, then $a \not\not b'$, and therefore there is a state $s \in S(L)$ such that $s(a) = 1$ and $s(b') \geq 1/2$. Since $s(b) = 1 - s(b')$, we see that $s(b) < 1/2$, as required.
It can be proved that all the inclusions in Theorem 48 are proper [Ptáček and Pulmannová, 1991]. The counter examples are, usually, constructed with the help of so called Greechie past job [Greechie, 1971; Greechie, 1969] or its generalizations [Mayet et al., 2000]. Let us present here a typical way of constructing interesting features of state spaces in this manner. Let us first introduce another class of logics ([Rogalewicz, 1984b] and [Ptáček and Rogalewicz, 1983b]). We point out that various other such classes having their origin in mathematical analysis have been studied [Binder and Navara, 1987; D'Andrea and de Lucia, 1991; De Simone and Navara, 2001; De Simone et al., to appear], etc.

**DEFINITION 51.** Let us denote by \( \mathcal{L}_{2_-} \) the following class of logics: \( L \in \mathcal{L}_{2_-} \iff \) if \( a \leftrightarrow b \) and if we are given an \( \varepsilon > 0 \) then there is a state \( s \in \mathcal{S}(L) \) such that \( s(a) \geq 1 - \varepsilon \) and \( s(b) \geq 1 - \varepsilon \).

The class \( \mathcal{L}_{2_-} \) is interesting in its own right but it is also relevant to the study of observables as we will see later.

**PROPOSITION 52.** \( \mathcal{L}_{2_-} \) is a proper subclass of \( \mathcal{L}_2 \).

**Proof.** We assume that the reader is to a certain extent familiar with the interpretation of the Greechie diagrams of a logic (see our next Figures 1, 2). Let us just recall that the “points” of a Greechie diagram are interpreted as the atoms of the logic and the straight line segments (or possibly the curve segments) group together those atoms which belong to a maximal orthogonal set (Boolean block). There is a one-to-one correspondence between the states on the logic constructed and the so called weights on the Greechie diagram. (A weight is such a (nonnegative) evaluation of the points of the diagram that the sums over Boolean blocks equal to 1.) We shall not make any distinction between the weights on a Greechie diagram and the states on the corresponding logic. The precise description of Greechie diagrams may be found in [Greechie, 1971] or [Schultz, 1974].

Let us return to the example disproving \( \mathcal{L}_2 \subseteq \mathcal{L}_{2_-} \). We start with a preliminary construction (Figure 1).

**Claim.** The logic \( L \) given by Figure 1 has the following properties:

1. \( L \in \mathcal{L}_{2_-} \),

2. If \( s \) is a state on \( L \) and if \( s(b) = 1 \), then \( s(a) = 0 \).

**Proof.** Suppose that \( s(b) = 1 \) for a state \( s \) on \( L \). Then \( s(f_i) = 0 \) for any \( i = 1, 2, 3, \ldots \). Therefore \( s(d) + \sum s(z_i) \leq s(d) + \sum s(c_i) \). Since \( s(e) = 0 \), the left side of the last inequality equals to one. Hence \( s(d) + \sum s(c_i) = 1 \) and therefore \( s(a) = 0 \). This was to prove. Observe that we have also proved that the identity \( s(b) = 1 \) for a state \( s \) implies \( s(a') = 1 \). On the other hand, if \( \varepsilon > 0 \) and \( r \in (0, 1) \) are given, we may find a state \( s \) such that \( s(b) = 1 - \varepsilon \) and \( s(a) = r \). It is easily seen that if we take two noncompatible elements \( p, q \in L \) such that \( (p, q) \neq (a, b) \), there exists a state \( s \) on \( L \) with \( s(p) = 1 = s(q) \). Hence \( L \in \mathcal{L}_{2_-} \).
Figure 1.

The desired example establishing $\mathcal{L}_{2-} \setminus \mathcal{L}_2 \neq \emptyset$ is now constructed as follows [Rogalewicz, 1984a].

**Claim.** The (lattice) logic $L$ given by Figure 2 belongs to $\mathcal{L}_{2-}$ but does not belong to $\mathcal{L}_2$.

**Proof.** If we take noncompatible atoms $(p, q) \neq (a, b)$, then one easily shows that there exists a state $s$ on $L$ such that $s(p) = 1 = s(q)$. Let us consider the pair $(a, b)$. If $s(b) = 1$, then $s(f_i) = 0$ for $i = 1, 2, 3, \ldots$ and therefore $s(d) + \sum s(c_i) = 1$. As a result, $s(a) \leq \frac{1}{2}$ and $L$ does not belong to $\mathcal{L}_{2-}$. On the other hand, $L$ belongs to $\mathcal{L}_{2-}$. Indeed, if we are given a $\varepsilon > 0$, we may simply construct a state $s$ with $s(b) = 1 - \varepsilon$ and $s(f_i) \in (0, \varepsilon)$.

An alternative proof of the previous result can be obtained by the following result, which uses the convexity of the state space.

**Proposition 53.** A logic $L$ belongs to $\mathcal{L}_{2-}$ if and only if it satisfies the following condition: If $a, b$ are two noncompatible elements of $L$ and if we are given a real number $r \in (0, 1)$ then there exists a state $s$ on $L$ such that $s(a) = r = s(b)$.

**Proof.** The condition is obviously sufficient. To prove necessity, let a real number $r < 1$ be given. We may suppose that $r \geq \frac{1}{2}$ — otherwise we take up the equivalent assertion with $a', b'$ and $r' = 1 - r$. We shall construct two states $s_1, s_2$ on $L$ such
that \( s_1(a) = s_1(b) = r_1 \in (0, \frac{1}{3}) \) and \( s_2(a) = s_2(b) = r_2 \in (r, 1) \). The required state \( s \) on \( L \) can then be constructed by setting

\[
  s = \frac{(r_2 - r)s_1 + (r - r_1)s_2}{r_2 - r_1}.
\]

We shall first define \( s_1 \). If there is a state \( s \) on \( L \) such that \( s(a') = 1 = s(b') \), we put \( s_1 = s \). If this is not the case, the assumption guarantees the existence of states \( s_3, s_4 \) on \( L \) with \( s_3(a') = 1, s_3(b') = r_3 < 1, s_4(a') = r_4 < 1, s_4(b') = 1 \). Let us put

\[
  s_1 = \frac{(1 - r_4)s_3 + (1 - r_3)s_4}{(1 - r_3) + (1 - r_4)}.
\]

Let us now consider the construction of \( s_2 \). If there is a state \( s \) on \( L \) with \( s(a) = 1 = s(b) \), we put \( s_2 = s \) and the proof is complete. If there is no such state,
we consider states \( s_5, s_6 \) and \( s_7 \) such that \( s_5(a) = 1, s_5(b) = r_5 < 1, s_6(a) = r_6 < 1, s_6(b) = 1 \) and, further, if we set \( q = \max(r, r_5, r_6) \), we require \( s_7(a) = r_7 > q, s_7(b) = r_8 > 2 \). The assumptions guarantee the existence of such states. If \( r_7 = r_s \), we put \( s_2 = s_7 \). If \( r_7 < r_s \), then

\[
s_2 = \frac{(r_s - r_7)s_5 + (1 - r_5)s_7}{(1 - r_5) + (r_s - r_7)}.
\]

If \( r_s < r_7 \), the construction of \( s_2 \) proceeds dually. Now the proof is complete. ■

**Definition 54.** An element \( a \in L \) is called a *support* of a state \( s \) on a logic \( L \), if for any \( b \in L \), \( a \perp b \) if and only if \( s(b) = 0 \). If \( a \) is a support of \( s \), we will write \( a = \text{supp} \ s \).

**Proposition 55.** Every Jauch-Piron state on a separable lattice logic has a support.

**Proof.** Let \( L \) be a separable lattice logic and let \( s \) be a state on \( L \). Put \( s^0 := \{ a \in L : s(a) = 0 \} \). By Proposition 22, the element \( a = \bigvee s^0 \) exists in \( L \), and \( a = \bigvee_{i \in \mathbb{N}} a_i \), where \( a_i \in s^0, i \in \mathbb{N} \). Put \( b_1 = a_1, b_2 = b_1 \lor a_2, \ldots, b_n = b_{n-1} \lor a_n, \ldots \). Clearly, \( (b_i)_i \) is a nondecreasing sequence with \( \bigvee_i b_i = a \). From the Jauch-Piron property we obtain \( s(b_i) = 0, i = 1, 2, \ldots \). By \( \sigma \)-additivity of \( s \), \( s(a) = s(\bigvee_i b_i) = \lim_{n \to \infty} s(b_i) = 0 \). We will show that \( \text{supp} \ s = a' \). If \( b \in L, s(b) = 0 \), then \( b \in s^0 \), and hence \( b \perp a' \). Conversely, if \( b \perp a' \), then \( b \leq a \), hence \( s(b) = 0 \). ■

The notion of a Jauch-Piron state can be strengthened further by introducing subadditive states (or so called valuations, see e.g. [Pták, 1998; Riečanová, 1998; Sarymsakov et al., 1983], etc.).

**Definition 56.** Let \( L \) be a logic. A state \( s \) on \( L \) is said to be subadditive if for any \( a, b \in L \) there exists \( c \in L \) such that \( a \leq c, b \leq c \) and \( s(c) \leq s(a) + s(b) \).

It is an easy exercise to show that each subadditive state is Jauch-Piron. Indeed, if \( s(a) = 1, s(b) = 1 \) then \( s(a') = 0, s(b') = 0 \). Since \( s \) is subadditive, there is \( c \in L \) such that \( a' \leq c, b' \leq c \) with \( s(c) \leq s(a') + s(b') = 0 \). Thus, \( s(c') = 1 \) and \( c' \leq a \) and \( c' \leq b \).

If a logic has "enough" states and if each state on it is subadditive, the logic is in a sense "almost Boolean". This situation is, however, fairly subtle as the following example shows (see also [Navara and Pták, 1988] and [Pták, 1998]).

**Example 57.** There is a concrete non-Boolean logic such that each state on it is subadditive.

Let us indicate the construction of such an example. Let \( P \) be a set of the first uncountable cardinality. Then each state on \( \exp P \) lives on a countable set a well-known result of measure theory (by \( \exp P \) we mean the Boolean \( \sigma \)-algebra of all subsets of \( P \)). In other words, if \( s \) is a state on \( \exp P \), then there is a countable subset, \( M \), of \( P \) such that \( s(M) = 1 \). Let \( L \) be the logic of all subsets
5 OBSERVABLES ON A LOGIC

DEFINITION 58. An observable on a logic \( L \) is a \( \sigma \)-homomorphism from a \( \sigma \)-algebra \( \Sigma \) of subsets of a set \( \Omega \) to \( L \), that is, a mapping \( x : \Sigma \rightarrow L \) is an observable if

(i) \( x(\Omega) = 1 \); 

(ii) \( x(E^c) = x(E)' \), \( E \in \Sigma \) (\( E^c \) is the complement of \( E \) in \( \Omega \)),

(iii) \( x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i) \) for every sequence \( (E_i)_i \) of pairwise disjoint subsets of \( \Sigma \).

The set \( \Omega \) is called the value space of the observable \( x \).

Let us notice that if \( (E_i)_i \) is any sequence of elements of \( \Sigma \), then \( x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i) \). This is obtained by replacing \( (E_i)_i \) by the sequence of pairwise disjoint sets \( (F_j)_j \), where \( F_1 = E_1, F_n = E_n \backslash (\bigcup_{i=1}^{n-1} F_i) \), and \( \bigcup_i E_i = \bigcup_j F_j \). This entails that the range \( \mathcal{R}(x) := \{x(E) : E \in \Sigma\} \) of an observable \( x \) is a Boolean sublogic of \( L \).

A family \( (x_\alpha)_\alpha \) of observables on \( L \) is called compatible if the ranges of all \( x_\alpha \) belong to the same Boolean sublogic of \( L \).

If \( \Omega = \mathbb{R} \) and \( \Sigma \) is a sub-\( \sigma \)-algebra of the Borel \( \sigma \)-algebra \( B(\mathbb{R}) \) of subsets of \( \mathbb{R} \), then an observable \( x : \Sigma \rightarrow L \) is called a real observable: In this case, if \( L \) is a concrete logic, then an observable is nothing but a classical random variable [Varadarajan, 1985], and this is the case in many other situations [Pták, 1981].

If \( f : \Omega \rightarrow \mathbb{R} \) is a measurable function and \( x : \Sigma \rightarrow L \) is an observable, then \( f(x) : B(\mathbb{R}) \rightarrow L, f(x)(E) = x(f^{-1}(E)) \) is also an observable, which is called the function \( f \) of the observable \( x \).
The theory of real observables on a logic was developed by Varadarajan [Varadarajan, 1985]. The corner stone is the celebrated Loomis-Sikorski theorem. The statements in [Varadarajan, 1985] are formulated for a lattice logic but if we replace pairwise compatibility by the stronger notion of compatibility, the statements hold for arbitrary logics as well.

THEOREM 59. (Loomis–Sikorski), [Varadarajan, 1985, Theorem 1.3] Let $B$ be a Boolean $\sigma$-algebra. Then there exists a measurable space $(\Omega, \Sigma)$, where $\Sigma$ is a $\sigma$-algebra of subsets of a nonempty set $\Omega$, and a $\sigma$-homomorphism $h$ from $\Sigma$ onto $B$.

THEOREM 60. [Varadarajan, 1985, Theorem 1.6, Lemma 3.16] Let $x$ be a real observable on a logic $L$. Then the range $R(x) = \{x(E) : E \in B(\mathbb{R})\}$ of $x$ is a countably generated Boolean sublogic of $L$. Conversely, if $B$ is a countably generated Boolean sublogic of $L$, then there exists a real observable $x$ such that $R(x) = B$.

THEOREM 61. [Varadarajan, 1985, Theorem 1.4] Let $(\Omega, \Sigma)$ be a measurable space, and let $h$ be a $\sigma$-homomorphism from $\sigma$ onto a Boolean $\sigma$-algebra $B$. Let $u$ be a real observable with range in $B$. Then there exists a $\Sigma$-measurable function $f : \Omega \to \mathbb{R}$ such that $u(E) = h(f^{-1}(E))$, $E \in B(\mathbb{R})$. The function $f$ is unique in the sense that if $g$ is any other function with the same properties, then $\{\omega \in \Omega : f(\omega) \neq g(\omega)\}$ belongs to the null space of $h$.

THEOREM 62. [Varadarajan, 1985, Theorem 3.9] Let $\{x_{\alpha} : \alpha \in A\}$ be an arbitrary family of compatible real observables on a logic $L$. Then there exists a set $X$, a $\sigma$-algebra $B$ of subsets of $X$, real-valued $B$-measurable functions $g_{\alpha}$ on $X(\alpha \in A)$, and a $\sigma$-homomorphism $\tau$ of $B$ into $L$ such that

$$\tau(g_{\alpha}^{-1}(E)) = x_{\alpha}(E)$$

for all $\alpha \in A$ and $E \in B(\mathbb{R})$.

Suppose further that either $L$ is separable, or that $A$ is countable. Then there exists an observable $x$ and real valued Borel functions $f_{\alpha}$ of a real variable such that $\forall \alpha \in A$, $x_{\alpha} = f_{\alpha} \circ x$.

Notice that for at most two observables the condition of compatibility in Theorem 62 may be replaced by pairwise compatibility [Ramsay, 1966].

Theorem 62 enables us to construct a calculus of functions of several compatible observables.

THEOREM 63. [Varadarajan, 1985, Theorem 3.17] Let $L$ be a logic and $x_1, x_2, \ldots, x_n$ compatible observables on $L$. Then there exists one and only one $\sigma$-homomorphism $\tau$ of $B(\mathbb{R}^n)$ into $L$ such that for all $E \in B(\mathbb{R})$ and all $i = 1, 2, \ldots, n$,

$$x_i(E) = \tau(\pi_i^{-1}(E)),$$

where $\pi_i$ is the projection $(t_1, t_2, \ldots, t_n) \mapsto t_i$ of $\mathbb{R}^n$ to $\mathbb{R}$. If $g$ is any real-valued Borel function on $\mathbb{R}^n$,

$$g \circ (x_1, x_2, \ldots, x_n) : E \to \tau(g^{-1}(E))$$
is an observable. If \( g_1, g_2, \ldots, g_k \) are real valued Borel functions on \( \mathbb{R}^n \) and \( y_i = g_i \circ (x_1, x_2, \ldots, x_n) \), then \( y_1, y_2, \ldots, y_k \) are compatible, and for any real valued Borel function \( h \) on \( \mathbb{R}^k \),
\[
h \circ (y_1, y_2, \ldots, y_k) = (h(g_1, g_2, \ldots, g_k)) \circ (x_1, x_2, \ldots, x_n),
\]
where \( h(g_1, g_2, \ldots, g_k) \) is the function
\[
t = (t_1, \ldots, t_n) \mapsto h(g_1(t), g_2(t), \ldots, g_k(t)).
\]

The \( \sigma \)-homomorphism \( \tau \) of Theorem 63 is called the joint observable of \( x_1, x_2, \ldots, x_n \).

In analogy to self-adjoint operators, we can also introduce the notions of an eigenvalue and the spectrum of a real observable.

**DEFINITION 64.** A real number \( t \) is an eigenvalue of a real observable \( x \) if \( x(\{t\}) \neq 0 \).

**DEFINITION 65.** Let \( x \) be a real observable on a logic \( L \). Put
\[
\sigma(x) := \bigcap \{ C : C \text{ is a closed subset of } \mathbb{R} \text{ with } x(C) = 1 \}.
\]
Then the closed set \( \sigma(x) \subset \mathbb{R} \) is called the spectrum of the observable \( x \).

**PROPOSITION 66.** The spectrum \( \sigma(x) \) of a real observable \( x \) is the smallest closed subset of \( \mathbb{R} \) such that \( x(C) = 1 \).

**Proof.** Since the topology of the real line \( \mathbb{R} \) satisfies the second countability axiom, there exists a sequence of closed subsets \( C_1, C_2, \ldots \) such that \( x(C_n) = 1 \) for each \( n \), and \( \sigma(x) = \bigcap_{n=1}^{\infty} C_n \). Since \( x(\mathbb{R} \setminus C_n) = 0 \) for all \( n \), we have \( x(\mathbb{R} \setminus \sigma(x)) = x(\bigcup_{n=1}^{\infty} \mathbb{R} \setminus C_n) = \bigvee_{n=1}^{\infty} x(\mathbb{R} \setminus C_n) = 0 \), which entails that \( x(\sigma(x)) = 1 \).

A number \( r \in \sigma(x) \) is a spectral value of \( x \). An eigenvalue is a spectral value, the converse need not hold.

**LEMMA 67.** A real number is a spectral value of an observable \( x \) if and only if \( x(U) \neq 0 \) for every open subset \( U \subset \mathbb{R} \) which contains \( r \).

**Proof.** Let \( U \) be an open set containing \( r \) and let \( x(U) = 0 \). Then \( \mathbb{R} \setminus U \) is closed and \( x(\mathbb{R} \setminus U) = 1 \). This yields \( \mathbb{R} \setminus U \supset \sigma(x) \), so that \( r \notin \sigma(x) \).

Conversely, assume that \( r \notin \sigma(x) \). Since \( \mathbb{R} \) is a metric space, there exist open subsets \( U_1, U_2 \) such that \( r \in U_1, \sigma(x) \subset U_2 \) and \( U_1 \cap U_2 = \emptyset \). Then \( x(U_2) = 1 \) and this implies that \( x(U_1) = 0 \).

**DEFINITION 68.** A real observable \( x \) is said to be

(i) **discrete** if \( \sigma(x) = \{ r_1, r_2, \ldots \} \);

(ii) **constant** if \( \sigma(x) = \{ r \} \);
(iii) simple if \( \sigma(x) = \{ r_1, r_2, \ldots, r_n \} \);

(iv) a proposition or \( 0 - 1 \) observable if \( \sigma(x) \subseteq \{0,1\} \).

For \( a \in L \), denote by \( q_a \) the unique \( 0 - 1 \) observable for which \( q_a(\{1\}) = a \). The observable corresponding to \( 0 \in L \), \( q_0 \), is the null observable. The observable corresponding to \( 1 \in L \), \( q_1 \), is the unit observable. The null and unit observables are constants, and \( r.q_1 \) (defined by the function calculus) is a constant observable with the spectrum \( \sigma(x) = \{r\} \).

**Lemma 69.** (i) For every \( a \in L \), \( q_a = f(q_a) \) with \( f(t) = 1 - t \) (\( t \in \mathbb{R} \)). (ii) If \( x \) is an observable and \( \chi_E \) is a characteristic function of a set \( E \in \mathcal{B}(\mathbb{R}) \), then \( \chi_E(x) = q_{\chi(E)} \).

**Proposition 70.** The following conditions are equivalent:

(i) \( x \) is a proposition observable;

(ii) there exists an observable \( y \) and \( E \in \mathcal{B}(\mathbb{R}) \) such that \( x = \chi_E(y) \);

(iii) \( x^2 = x \).

**Proof.** (i) \( \Rightarrow \) (ii): By Lemma 69, \( \chi_{\{1\}}(x) = x \). (ii) \( \Rightarrow \) (iii): If \( x = \chi_E(y) \), then \( x^2 = \chi_E^2(y) = \chi_E(y) = x \). (iii) \( \Rightarrow \) (i): Put \( f(t) = t, g(t) = t^2, t \in \mathbb{R} \). We have \( \{t : f(t) \neq g(t)\} \subseteq \bigcup_{r \in \mathbb{Q}} f^{-1}(\infty, r) \Delta g^{-1}(\infty, r) \), where \( \mathbb{Q} \) is the set of rational numbers, and \( E \Delta F = (E \cap F^c) \cup (E^c \cap F) \) is the symmetric difference of sets. Then \( x(f^{-1}(\infty, r) \Delta g^{-1}(\infty, r)) = 0 \), which entails that \( x(\{t : t^2 = t\}) = x(\{0,1\}) = 1 \), hence \( x \) is a proposition observable.

6. **PARTIAL COMPATIBILITY AND JOINT DISTRIBUTIONS OF OBSERVABLES**

In the sequel we will introduce the notion of partial compatibility and that of a commutator. We will then study properties of these concepts and relations between them. These results will be used in our study of joint distributions of observables. We mostly restrict ourselves to lattice logics.

The notion of a commutator of two elements in an orthomodular lattice \( L \) was introduced in [Marsden, 1970]. The main result was that the orthomodular ideal \( J \) generated by the commutators of all pairs of elements of \( L \) is the smallest orthomodular ideal such that the quotient \( L/J \) is a Boolean algebra. The commutator of a finite subset of an orthomodular lattice \( L \) has been used in [Beran, 1979; Beran, 1985; Bruns and Greechie, 1990; Bruns and Kalmbach, 1973; Greechie and Herman, 1985; Chevalier, 1984; Chevalier, 1989; Chevalier and Pulmannová, 2000; Kalmbach, 1983], and [Poguntke, 1983].

The generalization of the concept of a commutator to arbitrary subsets of an orthomodular lattice \( L \) was studied in a series of papers [Dvurečenskij and Pulmannová, 1982; Dvurečenskij and Pulmannová, 1984; Lutterová and Pulmannová,
1985; Pulmannová, 1980; Pulmannová, 1981b; Pulmannová, 1985; Dvurečenskij and Pulmannová, 1982], see also [Dvurečenskij, 1993; Pták and Pulmannová, 1991], the notion of partial compatibility was introduced in [Pulmannová, 1981b].

The notion of a joint probability distribution of Gudder type was introduced in [Gudder, 1968; Gudder, 1979]. The subsequent developments can be followed in [Dvurečenskij, 1982]—[Dvurečenskij, 1987b], [Lahti and Ylinen, 1987; Lutterová and Pulmannová, 1985; Pulmannová, 1978; Pulmannová, 1980; Pulmannová, 1981a; Pulmannová, 1985; Dvurečenskij and Pulmannová, 1982; Dvurečenskij and Pulmannová, 1984; Pulmannová and Dvurečenskij, 1985], and [Pulmannová and Stehlíková, 1986].

Heisenberg’s uncertainty relations were expressed in the language of quantum logics in [Lahti, 1980; Bugajski and Lahti, 1980; Lahti and Mączyński, 1987]. Results on the relation between uncertainty relations and total noncompatibility can be found in [Pulmannová and Dvurečenskij, 1985] The notion of complementarity was introduced in [Lahti, 1980]. There is a huge list of publications devoted to these problems, which are relevant to quantum theory see, e.g., [Gudder, 1979; Jammer, 1974; Jauch, 1968; Holevo, 1980; Mackey, 1963; von Neumann, 1932; Primas, 1981].

The existence of sums of observables has already been emphasized by von Neumann [von Neumann, 1932]. In some axiomatic approaches to quantum mechanics the existence of sums of bounded observables is postulated (e.g., in [Segal, 1947], and the C*-algebraic approach [Alfsen and Schulz, 1979; Emch, 1972; Haag and Kastler, 1964]). The logics with the property that every two bounded observables are summable were investigated in [Gudder, 1966; Rüttimann, 1985], and [Dvurečenskij and Pulmannová, 1980].

Joint distributions of Urbanik type were introduced in [Urbanik, 1961] and in [Varadarajan, 1985] for observables on a Hilbert space logic. In [Gudder, 1966] a generalization of this notion to observables on a larger class of logics was studied. For the further development see, e.g., [Ruymgaart, 1983; Urbanik, 1985; Dvurečenskij and Pulmannová, 1989; Pulmannová and Dvurečenskij, 1980].

6.1 Partial compatibility and the commutator

Let $L$ be a logic, $D = \{-1,1\}$, and for $a \in L$, let $a^{-1} = a'$, $a^1 = a$.

**DEFINITION 71.** We say that a subset $A$ of a logic $L$ is partially compatible with respect to an element $b \in L$ (in short, $A$ is p.c. $b$) if the following conditions are satisfied:

(i) $b \leftrightarrow A$, i.e., $b \leftrightarrow a$ for all $a \in A$;

(ii) the set $b \land A := \{b \land a : a \in A\}$ is a compatible subset of $L$.

Recall that for a subset $A \subset L$, $C(A) = \{b \in L : b \leftrightarrow A\}$ is a sublogic of $L$, where $b \leftrightarrow A$ means that $b \leftrightarrow a \forall a \in A$. It what follows, let us agree to write $CC(M)$ instead of more correct $C(C(M))$.  
THEOREM 72. If for \( a \in L \) a finite subset \( M \) of \( L \) is p.c. \( a \), then \( CC(M) \) is p.c. \( a \).

**Proof.** Let \( M = \{a_1, a_2, \ldots, a_n\} \) and let \( \{e_1, e_2, \ldots, e_k\} \) be a minimal orthogonal cover of \( M \wedge a \) in \([0,a]\). Since \( a \in CC(M) \), \( a \leftrightarrow b \) for all \( b \in CC(M) \). From \( e_i \leftrightarrow a_j \wedge a \) and \( e_i \leq a \) for \( j \leq n, i \leq k \), we easily derive that \( e_i \in CC(M) \). Hence \( b \leftrightarrow e_i, i \leq k \). Therefore \( \{b \wedge e_i, b' \wedge e_i\}_{i=1}^k \) is an orthogonal cover of the set \( \{a_1 \wedge a, a_2 \wedge a, \ldots, a_n \wedge a, b \wedge a\} \), which shows that \( M \cup \{b\} \) is p.c.a. By induction, we see that every finite subset \( \{b_1, b_2, \ldots, b_n\} \subset CC(M) \) is p.c.a, hence \( CC(M) \) is p.c. \( a \).

If \( A \subseteq L \) is p.c. \( a \), we can prove, using Zorn’s lemma, that there is a maximal p.c. \( a \) subset of \( L \) containing \( A \).

THEOREM 73. A subset \( Q \) of \( L \) which is a maximal p.c \( a \) is a sublogic of \( L \). If \( L \) is a lattice then \( Q \) is also a lattice.

**Proof.** Let \((a_i)_{i \in \mathbb{N}}\) be a sequence of pairwise orthogonal elements of \( Q \). We need to show that \( \bigvee_i a_i \in Q \). From \( a_i \leftrightarrow a \forall i \) we derive that \( a \leftrightarrow \bigvee_i a_i \) and \((\bigvee_i a_i) \wedge a = \bigvee_i a \wedge a_i \). The set \( Q \wedge a \) is compatible in \([0,a]\), hence it is contained in a Boolean sublogic of \([0,a]\), and this Boolean sublogic contains also \( \bigvee_i a \wedge a_i \). This entails that \( Q \cup \{\bigvee_i a_i\} \) is p.c. \( a \), and maximality of \( Q \) implies that \( \bigvee_i a_i \in Q \). If \( b \in Q \), then the Boolean subalgebra of \([0,a]\) containing \( b \wedge a \) contains also \( (b \wedge a)' \wedge a = b' \wedge a \), hence \( b' \in Q \). This proves that \( Q \) is a sublogic of \( L \).

If \( L \) is a lattice, then for \( b, c \in Q \), \( a \vee b \) exists in \( L \), and the relations \( a \leftrightarrow b, a \leftrightarrow c \) imply that \( a \leftrightarrow b \vee c \) (a lattice logic is regular), and \( a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \). The Boolean sublogic of \([0,a]\) that contains \( Q \wedge a \) then contains also \( (a \wedge b) \vee (a \wedge c) \), whence \( Q \cup \{b \vee c\} \) is p.c. \( a \). By the maximality of \( Q \), \( a \vee b \in Q \). We conclude that \( Q \) is a lattice.

THEOREM 74. Let \( F = \{a_1, a_2, \ldots, a_n\} \) be a finite subset of \( L \). If \( com(F) \) exists in \( L \), then (i) \( F \) is p.c. \( com(F) \); (ii) if \( a \in L \) is such that \( F \) is p.c. \( a \), then \( a \leq com(F) \).

**Proof.** (i) From the definition of the commutator it is clear that \( a_i \leftrightarrow com(F) \) for all \( i \leq n \). Put \( c := com(F) \), and consider the interval \([0,c]\). From \( F \leftrightarrow c \), and existence of \( com(F) \) in \( L \), we derive that \( com(a_1 \wedge c, \ldots, a_n \wedge c) \) in \([0,c]\) exists, and is equal to \( com(F) \wedge c = c \), hence the elements \( a_1 \wedge c, a_2 \wedge c, \ldots, a_n \wedge c \) are compatible. This proves that \( F \) is p.c. \( com(F) \).

(ii) The statement is trivial for \( a = 0 \). Assume that \( F \) is p.c. \( a \) for \( a \neq 0 \). Then \( F \wedge a \) is compatible in \([0,a]\), hence \( \bigvee_{d \in D^n}(a_1 \wedge a)^{d_1} \wedge \cdots \wedge (a_n \wedge a)^{d_n} = a \), where we put \((a_i \wedge a)^{d_i} = a_i \wedge a \) if \( d_i = 1 \) and \((a_i \wedge a)^{d_i} = a'_i \wedge a \) if \( d_i = -1 \). Hence \( \bigvee_{d \in D^n} a_1^{d_1} \wedge \cdots a_n^{d_n} \wedge a = a \), and as \( a \leftrightarrow F \), this implies that \( com(F) \wedge a = a \), hence \( a \leq com(F) \).
In what follows we will tacitly assume that $L$ is a lattice logic. Let us first extend the notion of a commutator to arbitrary subsets of $L$.

**DEFINITION 75.** Let $M$ be a subset of a lattice logic $L$. If the following element $\text{com}(M)$ exists,

$$\text{com}(M) := \bigwedge \{\text{com}(F) : F \text{ is a finite subset of } M\},$$

it is called the commutator of the set $M$.

**THEOREM 76.** Let $M$ be a subset of a lattice logic $L$. If $\text{com}(M)$ exists, then $M$ is p.c. $\text{com}(M)$.

**Proof.** It is easy to see that if $F_1 \subseteq F_2$, where $F_1, F_2$ are finite subsets of $M$, then $\text{com}(F_1) \geq \text{com}(F_2)$. Let $b \in M$ be an arbitrary element. We have

$$\text{com}(M) = \bigwedge_F \text{com}(F) = \bigwedge_F \text{com}(F \cup \{b\}),$$

and since $b \leftrightarrow \text{com}(F \cup \{b\})$ for all $F \subseteq M$, by Lemma 8 we have $b \leftrightarrow \text{com}(M)$. Hence $M \leftrightarrow \text{com}(M)$. Let $b_1, b_2 \in M$. An easy computation shows that $\text{com}(b_1 \wedge \text{com}(M), b_2 \wedge \text{com}(M)) = \text{com}(b_1, b_2) \wedge \text{com}(M) = \text{com}(M)$. This entails that $b_1 \wedge \text{com}(M)$ and $b_2 \wedge \text{com}(M)$ are compatible in $[0, \text{com}(M)]$. Since a lattice logic is regular, we obtain that $M$ is p.c. $\text{com}(M)$.

**COROLLARY 77.** Let $M \subseteq L$. If $\text{com}(M)$ exists, then for every $a \in L$, $M$ is p.c. $a$ if and only if $M \leftrightarrow a$ and $a \leq \text{com}(M)$.

**Proof.** If $M$ is p.c. $a$, then every finite subset $F$ of $M$ is p.c. $a$, so that $a \leq \text{com}(F)$ by Theorem 74, and consequently $a \leq \text{com}(M)$. Conversely, assume that $M \leftrightarrow a$ and $a \leftrightarrow \text{com}(M)$. Let $b, c \in M$. Then the relations $b \wedge \text{com}(M) \leftrightarrow c \wedge \text{com}(M)$, $b \wedge \text{com}(M) \leftrightarrow a$, $c \wedge \text{com}(M) \leftrightarrow a$ imply that $b \wedge a \wedge \text{com}(M) \leftrightarrow c \wedge a \wedge \text{com}(M)$. Now if $a \leq \text{com}(M)$, then we obtain $b \wedge \text{com}(M) \leftrightarrow c \wedge \text{com}(M)$, i.e., $M$ is p.c. $a$.

**COROLLARY 78.** Let $A$ be a subset of $L$ such that $\text{com}(A)$ exists. If $L(A)$ is the lattice sublogic of $L$ generated by $A$, then $\text{com}(A) = \text{com}(L(A))$.

**Proof.** Observe that $L(A)$ is p.c. $\text{com}(A)$. Indeed, by Theorem 73, the maximal subset $Q$ of $L$ which contains $A$ and is p.c. $\text{com}(A)$, is a lattice sublogic of $L$. The result then follows from $L(A) \subseteq Q$. By Corollary 77, $\text{com}(A) \leq \text{com}(F)$ for every finite set $F \subseteq L(A)$. Let $c \in L$ be an element such that $c \leq \text{com}(F)$ for every finite $F \subseteq L(A)$. Then $c \leq \text{com}(G)$ for every finite $G \subseteq A$. Hence $c \leq \text{com}(A)$. This proves that

$$\text{com}(A) = \bigwedge \{\text{com}(F) : F \text{ is a finite subset of } L(A)\} = \text{com}(L(A)).$$
In what follows, we will often make use of the following important theorem, which extends the Foulis-Holland theorem [Greechie, 1979], see also [Kalmbach, 1983; Navara and Hýčko, to appear] and [Pták and Pulmannová, 1991, Prop. 5.1.13.a].

**THEOREM 79.** Let \( L \) be a lattice logic and let \( S \subset L \) have the property that among any three elements of \( S \) there is one which is compatible with the other two. Then the sublattice \( [S] \) of \( L \) generated by the set \( S \) is distributive (i.e., for any three elements \( a, b, c \in [S] \) we have \( a \land (b \lor c) = (a \land b) \lor (a \land c) \)).

In the next part we will use the following conventions. We will write \( \text{com}(M_1, M_2, \ldots, M_n) \) instead of \( \text{com}(M_1 \cup M_2 \cup \cdots \cup M_n) \), and sometimes we will use the symbol \( \text{com}(a_1, a_2, \ldots, a_n, Q) \) instead of the more correct notation of \( \text{com}([\{a_1, a_2, \ldots, a_n\}, Q]) \).

**THEOREM 80.** Let \( A_i \) (\( i \in \mathbb{N} \)) be finite subsets of mutually orthogonal elements in \( L \). Then for any finite subset \( Q \) of \( L \) we have

\[
\text{com}(A_1, A_2, \ldots, A_n, Q) = \bigwedge \{ \text{com}(a_1, a_2, \ldots, a_n, Q) : (a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n \}.
\]

**Proof.** We will proceed by induction. Assume that \( n = 1 \) and \( A_1 = \{a_1, a_2\} \). We will show that

\[
(2) \quad \text{com}(a_1, a_2, b_2, \ldots, b_n) = \text{com}(a_1, b_2, \ldots, b_n) \land \text{com}(a_2, b_2, \ldots, b_n).
\]

Consider the set

\[
B = \{ a_1 \land b^d, a_2 \land b^d, a'_1 \land b^d, a'_2 \land b^d : d \in D^{n-1} \},
\]

where \( b^d = b_2 \land \cdots \land b_{n}^d \). The set \( B \) contains four-element classes for different \( d \in D^{n-1} \). Any two elements from different classes are orthogonal. Any two elements in the same class are mutually compatible, with the possible exception of \( a'_1 \land b^d \) and \( a'_2 \land b^d \). Consequently, among any three elements of \( B \) there is always one compatible with the other two. Therefore we may use distributivity to obtain (2).

The proof then continues by induction, first with respect to the cardinality of \( A_1 \), then with respect to the number of sets \( A_i, i \leq n \). For details, see [Pták and Pulmannová, 1991, Prop. 5.1.14].

In the following results we will apply properties of commutators to observables. We will view observables as logic morphisms \( x : B(M) \to L \), where \( B(M) \) is the \( \sigma \)-algebra of Borel subsets of a separable Banach space \( M \) and \( L \) is a (lattice) logic. We will write \( R(x) := \{ x(E) : E \in B(M) \} \) for the range of the observable \( x \). Let us recall that \( R(x) \) is a Boolean sublogic of \( L \).

Let \( \{ x_\alpha : \alpha \in I \} \) be a system of observables. Let us write

\[
\text{com}\{x_\alpha : \alpha \in I\} := \text{com}(\bigcup_{\alpha \in I} R(x_\alpha)).
\]
If \( \text{com}\{x_\alpha : \alpha \in I\} \) exists, it is called the **commutator** of the system \( \{x_\alpha : \alpha \in I\} \) of observables.

In what follows, we will need the following result (see [Pták and Pulmannová, 1991, Theorem 5.1.17]).

**THEOREM 81.** Let \( M \) be a separable Banach space and let \( L \) be a logic. A Boolean sublogic \( B \) of \( L \) is the range of an observable \( x : B(M) \to L \) if and only if \( B \) is countably generated.

**THEOREM 82.** Let \( I \) be an at most countable set and let \( \{x_i : i \in I\} \) be a system of observables on a lattice logic \( L \). Then \( \text{com}\{x_i : i \in I\} \) exists in \( L \). Moreover, there is a countable set \( Q \) such that \( Q \subset \bigcup_{i \in I} R(x_i) \) such that \( \text{com}\{x_i : i \in I\} = \text{com}(Q) \).

**Proof.** Let \( Q_i \) denote a countable generator of \( R(x_i) \). Let \( L_i \) be the lattice sublogic of \( L \) generated by \( Q_i \). It is easy to see that \( L_i \subset R(x_i) \). Therefore \( L_i \) is compatible, hence a Boolean sublogic of \( L \), and therefore \( L_i = R(x_i) \). Further set \( Q = \bigcup_{i \in I} Q_i \). Evidently, \( Q \) is at most countable. Since the set of all finite subsets of a countable set is countable, and since \( L \) is a \( \sigma \)-lattice, the element \( \text{com}(Q) = \bigcap \{\text{com}(F) : F \text{ a finite subset of } Q\} \) exists in \( L \). Let \( L(Q) \) denote the lattice sublogic of \( L \) generated by \( Q \). By Corollary 78 \( \text{com}(Q) = \text{com}(L(Q)) \).

By the first part of the proof we have \( R(x_i) \subset L(Q) \) \( \lor \in I \). This implies that \( \text{com}(F) \geq \text{com}(Q) \) for every finite subset \( F \subset \bigcup_{i \in I} R(x_i) \). Let \( c \in L \) be any element such that \( c \leq \text{com}(F) \) for every finite subset \( F \) of \( \bigcup_{i \in I} R(x_i) \). Then \( \text{com}(G) \geq c \) for every finite subset \( G \subset Q \) (i.e., \( \text{com}(Q) \geq c \)). Hence \( \text{com}(Q) = \text{com}\{x_i : i \in I\} \).

**COROLLARY 83.** Let \( x_1, x_2, \ldots, x_n \) be observables on \( L \). Then

\[
\text{com}\{x_1, x_2, \ldots, x_n\} = \bigwedge \{\text{com}\{x_1(E_1), \ldots, x_n(E_n) : E_1, \ldots, E_n \in B(M)\} \}.
\]

Moreover, there exists a countable set \( \{E_1^i, \ldots, E_n^i : i \in I\} \) such that

\[
\text{com}\{x_1, x_2, \ldots, x_n\} = \bigwedge_{i \in I} \text{com}\{x_1(E_1^i), \ldots, x_n(E_n^i)\}.
\]

**Proof.** Observe that if \( a_1, a_2, \ldots, a_n \) are compatible elements of \( L \) and \( F \) is a finite subset of \( L \), then \( \text{com}\{a_1, a_2, \ldots, a_n, F\} = \text{com}(\{a_1^{d_1} \land \cdots \land a_n^{d_n} : d \in D^n\}, F) \). We see that if \( G \) is a finite subset of \( \bigcup_{i \in I} R(x_i) \), then \( G = G_1 \cup \cdots \cup G_n \), where \( G_i \subset R(x_i) \) \( \lor i \leq n \) are orthogonal sets. This implies that for every such finite set \( F \) we have \( \text{com}(F) = \bigwedge_j \text{com}\{x_1(E_1^j), \ldots, x_n(E_n^j)\} \), where the index \( j \) runs over a finite set. By Theorem 82, there is a countable system \( \{F_n\}_{n \in \mathbb{N}} \) such that \( \text{com}\{x_1, x_2, \ldots, x_n\} = \bigwedge_{n \in \mathbb{N}} \text{com}(F_n) \).

In what follows, we will use the expression "observables are partially compatible" if the union of ranges of the observables is partially compatible. In what follows we will often make use of the following proposition the proof of which is straightforward.
PROPOSITION 84. Suppose that \( a \neq 0 \) and suppose that \( \{ x_\alpha : \alpha \in I \} \) is a system of observables which are partially compatible with respect to \( a \). Then the mappings \( x_\alpha \land a : B(M) \to L[0,a] \) defined via \((x_\alpha \land a)(E) = x_\alpha(E) \land a\) are compatible observables on the logic \( L[0,a] \).

6.2 Joint probability distribution of Gunder type

In the following definition, we consider observables on the Borel \( \sigma \)-algebra \( B(M) \) of a separable Banach space \( M \). Notice that the functional calculus for real compatible observables (Theorem 63) can be extended also to the latter case ([Pták and Pulmannová, 1991, Chap. 4]) in view of the fact that all separable Banach spaces have isomorphic algebras of Borel sets. Throughout this paragraph, let us assume that \( L \) is a lattice logic.

DEFINITION 85. We say that the observables \( x_1, x_2, \ldots, x_n \) on \( L \) have a joint probability distribution of Gunder type in a state \( s \) if there is a (probability) state \( P_{x_1,\ldots,x_n}^s \) on \( B(M) \) such that, for every rectangle \( E_1 \times \cdots \times E_n \in B(M^n) \), the following equality is satisfied:

\[
P_{x_1,\ldots,x_n}^s(E_1 \times \cdots \times E_n) = s(x_1(E_1) \land \cdots \land x_n(E_n)).
\]

If such a state \( P_{x_1,\ldots,x_n}^s \) exists, we will call it the probability distribution of Gunder type of the observables \( x_1, \ldots, x_n \).

Since a state on \( B(M^n) \) is uniquely defined by its values on the rectangle sets, the joint probability distribution of Gunder type is uniquely defined whenever it exists.

The notion of joint probability distribution can be generalized in the following natural way. We say that a system \( \mathcal{O} \) of observables \( \{ x_\alpha : \alpha \in J \} \) has a joint distribution of Gunder type in a state \( s \) if the joint probability distribution of Gunder type in the state \( s \) exists for any finite subsystem \( x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n} \) \((\alpha_1, \alpha_2, \ldots, \alpha_n \in J) \) of \( \mathcal{O} \).

THEOREM 86. The joint distribution of Gunder type for compatible observables \( x_1, \ldots, x_n \) exists in any state \( s \) on \( L \).

**Proof.** Let \( x_1, x_2, \ldots, x_n \) be compatible observables. By the functional calculus (Theorem 63), there exist an observable \( z \) and measurable functions \( f_i : M \to M(i \leq n) \) such that \( x_i = z.f_i^{-1} \). Let \( f : M \to M^n \) be the mapping \( f(m) = (f_1(m), f_2(m), \ldots, f_n(m)) \). Then \( w := z.f^{-1}, w : B(M^n) \to L \) is the joint observable for \( x_1, \ldots, x_n \), and \( x_i = w.\pi_i^{-1} \), where \( \pi_i : M^n \to M \) is the projection. Let \( s \) be a state on \( L \). Then \( s.w : B(M^n) \to [0,1] \) is a probability measure on \( B(M^n) \). Moreover, for every \( E_1 \times E_2 \times \cdots \times E_n \in B(M^n) \) we have

\[
s.w(E_1 \times \cdots \times E_n) = s(z.f^{-1}(E_1 \times \cdots \times E_n))
\]

\[
= s(z.((\pi_1.f)^{-1}(E_1) \cap \cdots \cap (\pi_n.f)^{-1}(E_n)))
\]

\[
= s(x_1(E_1) \land \cdots \land x_n(E_n)).
\]
It follows that $s.w$ is the joint distribution of Gudder type of $x_1, \ldots, x_n$ in the state $s$.

As a corollary we obtain that the joint distribution of Gudder type exists for an arbitrary system of compatible observables. The following theorem gives a general criterion of the existence of the joint distribution of Gudder type in a given state $s$.

**THEOREM 87.** Let $J$ be an arbitrary set. The observables $\{x_\alpha : \alpha \in J\}$ have a joint probability distribution in a state $s$ if and only if for every $\alpha_1, \ldots, \alpha_n \in J$ and every $E_1, \ldots, E_n \in \mathcal{B}(M)$ the following equality holds:

\[
(3) \quad s(\text{com}\{x_{\alpha_1}(E_1), x_{\alpha_2}(E_2), \ldots, x_{\alpha_n}(E_n)\}) = 1.
\]

**Proof.** Assume that equation (3) holds for every $\alpha_1, \ldots, \alpha_n \in J$ and every $E_1, \ldots, E_n \in \mathcal{B}(M)$. Fix $\alpha_1, \alpha_2, \ldots, \alpha_n \in J$ and put $x_{\alpha_i} = x_i, i = 1, 2, \ldots, n$. By Corollary 83, there is a countable set $\{E_i^1, \ldots, E_i^n : i \in I\}$ such that

\[
\text{com}(x_1, x_2, \ldots, x_n) = \bigcap_{i \in I} \text{com}(x_1(E_i^1), \ldots, x_n(E_i^n)).
\]

By (3), $s(\text{com}\{x_1(E_i^1), \ldots, x_n(E_i^n)\}) = 1$ for all $i \in I$. The $\sigma$-additivity of $s$ implies that $s(\text{com}\{x_1, x_2, \ldots, x_n\}) = 1$. Write $c := \text{com}\{x_1, x_2, \ldots, x_n\}$. Then $x_i, i = 1, 2, \ldots, n$, are partially compatible with respect to $c$ and by Proposition 84, $x_i \land c, i = 1, 2, \ldots, n$ are compatible observables on the logic $[0, c]$. Since $s(c) = 1$, the restriction of $s$ to $[0, c]$ is a state on $[0, c]$. It follows that the observables $x_1 \land c, x_2 \land c, \ldots, x_n \land c$ have a joint distribution of Gudder type in the state $s$, and hence there is a measure $\mu$ on $\mathcal{B}(M^n)$ such that for every rectangle $E_1 \times \cdots \times E_n$,

\[
\mu(E_1 \times \cdots \times E_n) = s(x_1 \land c(E_1) \land \cdots \land x_n \land c(E_n)).
\]

Let $L_0$ be the lattice sublogic of $L$ generated by $\bigcup_{i \leq n} R(x_i)$. By Corollary 78, $\text{com}(L_0) = c$, and $L_0$ is p.c. $c$. The restriction of $s$ to $L_0$ is clearly a state on $L_0$. Therefore for every $\alpha \in L_0, a = a \land c \lor a \land c'$. Further, $s(c) = 1$ implies that $s(a) = s(a \land c)$. This entails that

\[
\mu(E_1 \times \cdots \times E_n) = s(x_1(E_1) \land x_2(E_2) \land \cdots \land x_n(E_n) \land c)
\]

\[
= s(x_1(E_1) \land x_2(E_2) \land \cdots \land x_n(E_n)),
\]

hence the joint distribution of $x_1, \ldots, x_n$ in the state $s$ exists. Since the choice of $\alpha_1, \ldots, \alpha_n$ was arbitrary, the system $\{x_\alpha : \alpha \in J\}$ has a joint distribution.

Conversely, assume that the joint distribution of Gudder type for observables $\{x_\alpha : \alpha \in J\}$ exists in state $s$. Let $x_1, x_2, \ldots, x_n$ be arbitrary finite subsystem of the system $\{x_\alpha : \alpha \in J\}$. Then there is a probability measure $\mu$ on $\mathcal{B}(M^n)$ such that $\mu(E_1 \times \cdots \times E_n) = s(x_1(E_1) \land \cdots \land x_n(E_n))$ for every $E_1, \ldots, E_n \in \mathcal{B}(M)$. From the properties of a measure $\mu$ it can be easily derived that

\[
\text{com}(x_1(E_1), x_2(E_2), \ldots x_n(E_n)) = \sum_{d \in \mathcal{D}_n} s(\bigwedge_{i \leq n} x_i(E_i)^{d_i}) = 1.
\]
Recall that by Theorem 82, the commutator \( \text{com}\{x_i : i \in I\} \) exists for every at most countable system of observables \( \{x_i : i \in I\} \). Using properties of the commutator, we can easily derive the following result.

**THEOREM 88.** Let \( I \) be an at most countable set. A system of observables \( \{x_i : i \in I\} \) has a joint distribution in a state \( s \) if and only if \( s(\text{com}\{x_i : i \in I\}) = 1 \).

Let us note that if the joint distribution of Gudder type for observables \( \{x_i : i \in I\} \) exists in the state \( s \), then the statistical properties of these observables with respect to the state \( s \) are the same as the properties of the compatible observables \( \{x_i \land c : i \in I\} \), where \( c = \text{com}\{x_i : i \in I\} \). We may say that the existence of a joint distribution enables one to "compatibly reduce" the observables.

In the next theorem, we introduce a criterion which does not depend on the notion of a commutator.

**THEOREM 89.** The observables \( x_1, x_2, \ldots, x_n \) have a joint distribution of Gudder type in state \( s \) if and only if

\[
s(x_i(E_1^i \cup E_2^i) \land \bigwedge_{j \leq n, j \neq i} x_j(E_j)) = s(x_i(E_1^i) \land \bigwedge_{j \leq n, j \neq i} x_j(E_j)) + s(x_i(E_2^i) \land \bigwedge_{j \leq n, j \neq i} x_j(E_j)) \tag{4}
\]

for all \( i \leq n \) and \( E_1^i, E_2^i, E_j \in \mathcal{B}(\mathbb{R}) \) such that \( E_1^i \cap E_2^i = \emptyset \).

**Proof.** We sketch the proof for \( n = 2 \), for \( n > 2 \) the proof is analogous. Let \( \mathcal{D} \) be the class of all measurable rectangles \( E \times F \in \mathcal{B}(\mathbb{R}^2) \) and let \( \mathcal{A} \) be the algebra of all finite disjoint unions of elements in \( \mathcal{D} \). For \( E = E_1 \times E_2 \in \mathcal{D} \), define a set function \( \mu_0 \) by putting

\[
\mu_0(E) = s(x_1(E_1) \land x_2(E_2)).
\]

Using condition (4) it can be proved that \( \mu_0 \) is finitely additive on \( \mathcal{D} \). Let \( E \in \mathcal{A} \), \( E = \bigcup_{i \leq n} E_i, E_i \in \mathcal{D}, E_i \cap E_j = \emptyset, i, j \in \{1, 2, \ldots, n\} \). Put

\[
\mu_1(E) = \sum_{i \leq n} \mu_0(E_i).
\]

Then the set function \( \mu_1 : \mathcal{A} \to [0, 1] \) is well defined and finitely additive. By a well known theorems of classical measure theory (e.g., [Halms, 1962; Pták and Pulmannová, 1991]), \( \mu_1 \) is \( \sigma \)-additive, and hence it has a unique extension to a \( \sigma \)-additive measure \( \mu \) on \( \mathcal{B}(\mathbb{R}^2) \). Thus, \( \mu \) is the joint distribution of Gudder type of \( x_1, x_2, \ldots, x_n \) in the state \( s \).

In the following definition, we introduce the concept of "degree" of noncompatibility for a countable system of observables.

**DEFINITION 90.** Let \( I \) be an at most countable set. We say that the observables \( \{x_i : i \in I\} \) are:
(i) compatible if \( \text{com}\{x_i : i \in I\} = 1 \);

(ii) partially compatible if \( 0 < \text{com}\{x_i : i \in I\} < 1 \);

(iii) totally noncompatible if \( \text{com}\{x_i : i \in I\} = 0 \).

6.3 Uncertainty relations and complementarity

In this section we introduce a generalized definition of the famous Heisenberg uncertainty relations and a definition of complementarity for real observables on a unital lattice logic. We establish connections between concepts studied previously and commutators of observables.

Recall that a set \( S \) of states on a logic \( L \) is called unital if for every \( a \in L \), \( a \neq 0 \), there is a state \( s \in S \) with \( s(a) = 1 \). Note that the previous results allow us to formulate the following result, which is a sort of "classification" of observables.

PROPOSITION 91. Let \( I \) be an at most countable set and let \( S \) be a unital set of states on \( L \). The observables \( \{x_i : i \in I\} \) on \( L \) are

(i) compatible if and only if the joint distribution of Gudder type exists for all states \( s \in S \).

(ii) partially compatible if and only if there are states \( s_1, s_2 \in S \) such that the joint distribution of Gudder type exists in \( s_1 \) but does not exist in \( s_2 \).

(iii) totally noncompatible if and only if the joint distribution of Gudder type does not exist in each state \( s \in S \).

The proof is obvious from Theorem 88.

In the sequel, let us consider real observables on a lattice logic \( L \) with a unital set of states \( S \). The expectation of a real observable \( x \) in a state \( s \) is defined by

\[
(5) \quad s(x) = \int_\mathbb{R} t s(x(dt)),
\]

provided the integral on the right exists.

Let us first mention a problem which was popular in the eighties and nineties: the problem of uniqueness of observables (see [Gudder, 1979; Gudder, 1966], etc.). This problem is still not completely solved. It reads as follows: Suppose that for two bounded observables \( x, y \) we have \( s(x) = s(y) \) for any state \( s \in S(L) \), does it follow that \( x = y \)? This question, which is very important for quantum stochastics, has the answer yes for projection logics as well as for Boolean logics, but in general the answer is no, (see [Navara, 1995] for a counterexample and review). Let us here present one of the positive results in order to gain insight into the problem. We will show that the question has a positive answer for the logics of the class \( \mathcal{L}_2 \) considered in section 4 (we will recall the definition in the formulation of the result).

THEOREM 92. Let \( L \) be a logic such that
(i) for any $a \in L$ there is a state $s \in S$ such that $s(a) = 1$ (i.e., $L$ is unital),

(ii) for any $\varepsilon \in (0, 1)$ and any $a, b \in L$ with $a \neq b$ there is a state $s \in S$ such that $s(a) \geq 1 - \varepsilon$, $s(b) \geq 1 - \varepsilon$.

Let $x, y$ be two bounded observables on $L$ such that $s(x) = s(y)$ for any $s \in S$. Then $x = y$.

Let us first prove a lemma.

**Lemma 93.** Let $L$ be a logic such that for any $a \in L$ there is a state $s$ on $L$ such that $s(a) = 1$. Let $x, y$ be two observables on $L$. If $s(x) = s(y)$ for any $s \in S$, then for any real $r$ the relation $x(r, +\infty) \leftrightarrow y(r, +\infty)$ implies $x(r, +\infty) = y(r, +\infty)$.

**Proof.** Let $x, y$ be two observables and let $r$ be a real number. Put $a = x(r, +\infty)$, $b = y(r, +\infty)$ and suppose that $a \leftrightarrow b$. Then there are elements $a_1, b_1, c \in L$ such that $a_1 \leq b_1$, $a_1 \leq c$, $b_1 \leq c'$ and $a = a_1 + c$, $b = b_1 + c$. Let $a_1 \neq 0$. Then there is a state $s$ on $L$ with $s(a_1) = 1$. It follows that $s(b_1) = 0 = s(c)$ and thus $s(a) = 1$, $s(b) = 0$, $s(b') = 1$. Hence $s(x) \geq r$, $s(y) < r$ — a contradiction. Analogously, $b_1 \neq 0$ is not possible and we get $a = c = b$. This completes the proof.

Let us come to the proof of Th. 92. Suppose that $x \neq y$. By the preceding lemma, there exists an $r \in R$ such that $x(r, +\infty) \not\leftrightarrow y(r, +\infty)$. Hence $x(-\infty, r) \not\leftrightarrow y(r, +\infty)$. Since $\bigvee_{n=1}^{\infty} x(-\infty, r - \frac{1}{n}) = x(-\infty, r)$, there exists a number $n \in N$ such that $x(-\infty, r - \frac{1}{n}) \not\leftrightarrow y(r, +\infty)$. Choose real numbers $K, \varepsilon$ such that $y(-K, K) = 1$ and $\varepsilon(2nK + 1) < 1$. Take a state $s \in S$ that satisfies

$$s[x(-\infty, r - \frac{1}{n})] \geq 1 - \varepsilon, \quad s[y(r, +\infty)] \geq 1 - \varepsilon.$$ 

Then

$$s(x) < (1 - \varepsilon) \cdot (r - \frac{1}{n}) + K\varepsilon = (1 - \varepsilon)r - \frac{(1 - \varepsilon)}{n} + K\varepsilon,$$

$$s(y) \geq (1 - \varepsilon)r - K\varepsilon$$

and

$$s(y) - s(x) > -K\varepsilon - K\varepsilon + \frac{(1 - \varepsilon)}{n} = \frac{1}{n} - \varepsilon \left(\frac{1}{n} + 2K\right) = \frac{[1 - \varepsilon(1 + 2nK)]}{n} > 0.$$ 

Thus $s(y) > s(x)$, which is a contradiction. This completes the proof.

The variance of $x$ in the state $s$ is defined by

$$var_s(x) = \int_{\mathbb{R}} (t - s(x))^2 s(x(dt)),$$

provided the integral on the right exists.

For observables $x_1, x_2, \ldots, x_n$ we put
(7) \( V(x_1, \ldots, x_n) = \{ s \in S : \text{var}_s(x_i) < \infty, i \leq n \} \).

The following definition can be considered an expression of Heisenberg's uncertainty relations in the logico-algebraic approach.

**DEFINITION 94.** Let \( x_1, x_2, \ldots, x_n \) be real observables on \( L \). Then the following two cases can occur:

1. \((\exists \epsilon > 0)(\forall s \in V(x_1, \ldots, x_n)) : \text{var}_s(x_1)\text{var}_s(x_2)\ldots\text{var}_s(x_n) \geq \epsilon; \)
2. \((\forall \epsilon > 0)(\exists s \in V(x_1, \ldots, x_n)) : \text{var}_s(x_1)\text{var}_s(x_2)\ldots\text{var}_s(x_n) < \epsilon. \)

In the case (1) we say that the observables \( x_1, \ldots, x_n \) satisfy the uncertainty relation. In the case (2) we say that the observables \( x_1, \ldots, x_n \) do not satisfy the uncertainty relation.

**PROPOSITION 95.** If the observables \( x_1, \ldots, x_n \) are compatible, they do not satisfy the uncertainty relation.

**Proof.** Let \( w : B(M) \to L \) be the joint observable for \( x_1, \ldots, x_n \). By Lemma 67 (generalized in an obvious way to observables on \( B(\mathbb{R}^n) \)), a point \((r_1, r_2, \ldots, r_n)\) belongs to the spectrum \( \omega(w) \) of the observable \( w \) iff for any \( \eta > 0 \) we have

\[
\omega((r_1 - \eta, r_1 + \eta) \times \cdots \times (r_n - \eta, r_n + \eta)) \neq 0.
\]

Since \( S \) is unital, there is a state \( s \in S \) such that

\[
s(w(r_1 - \eta, r_1 + \eta) \times \cdots \times (r_n - \eta, r_n + \eta)) = 1.
\]

The properties of the joint observable imply that

\[
s(x_i((r_i - \eta, r_i + \eta))) = 1 \ (i \leq n).
\]

This yields, for all \( i \ (i \leq n) \),

\[
\text{var}_s(x_i) = \int_{r_i - \eta}^{r_i + \eta} (r - s(x_i))^2 s(x_i(dr)) \leq 4\eta^2.
\]

The last inequality implies that the observables \( x_1, \ldots, x_n \) do not satisfy the uncertainty relation. \[\blacksquare\]

**THEOREM 96.** If the observables \( x_1, \ldots, x_n \) satisfy the uncertainty relation, then \( \text{com}(x_1, \ldots, x_n) = 0 \).

**Proof.** Put \( c = \text{com}(x_1, \ldots, x_n) \) and suppose that \( c \neq 0 \). Define \( S_c := \{ s \in S : s(c) = 1 \} \). Since \( S \) is unital, \( S_c \neq 0 \). If we restrict \( S_c \) to the logic \([0, c]\), we easily verify that \( S_c \) is a unital set of states for \([0, c]\). The observables \( x_i \wedge c \ (i \leq n) \) are compatible and have the same probability distributions in every state \( s \in S_c \) as the observables \( x_i \ (i \leq n) \). By Proposition 95, the observables \( x_1 \wedge c, \ldots, x_n \wedge c \) on \([0, c]\) do not satisfy the uncertainty relation. Since the variances of the observables \( x_i \wedge c \ (i \leq n) \) are equal to the variances of the observables \( x_i \ (i \leq n) \) for all states in \( S_c \), the latter observables do not satisfy the uncertainty relation. This concludes the proof. \[\blacksquare\]
The following definition is an attempt to formulate the notion of complementarity in the frame of quantum logics. It is motivated by the notion of complementarity of observables on the Hilbert space logic \( L(H) \). On \( L(H) \) the observables \( P \) and \( Q \) corresponding to momentum and position, respectively, are complementary.

**Definition 97.** We say that two observables \( x \) and \( y \) are complementary if for any bounded sets \( E, F \) in \( B(\mathbb{R}) \) such that \( \omega(x) \not\subseteq E, \omega(y) \not\subseteq F \) we have \( x(E) \land y(F) = 0 \).

In the following propositions we list elementary facts concerning complementarity. The simple proofs are left to the reader.

**Proposition 98.** Two compatible observables are complementary if and only if at least one of them is a constant.

**Proposition 99.** Let \( a, b \ (a, b \neq 0, 1) \) be two elements of \( L \) and let \( q_a, q_b \) be the corresponding proposition observables such that \( q_a(\{1\}) = a, q_b(\{1\}) = b \). Then \( q_a \) and \( q_b \) are complementary if and only if they are totally noncompatible.

**Proposition 100.** If the observables \( x \) and \( y \) are complementary and noncompatible, then they are totally noncompatible.

The following example shows that the converse of Proposition 100 does not hold. Consider the Hilbert space \( \mathbb{R}^3 \). Notice that there exists no pair of nontrivial complementary observables. Let \( B_1 = \{e_1, e_2, e_3\} \) and \( B_2 = \{f_1, f_2, f_3\} \) be two bases of \( \mathbb{R}^3 \) with \( B_1 \cap B_2 = \emptyset \). Let \([e]\) denote the one-dimensional subspace generated by \( e \in \mathbb{R}^3 \). Define two observables \( x \) and \( y \) on \( L(\mathbb{R}^3) \) as follows. Choose three real numbers \( r_1, r_2, r_3 \) and put \( x(\{r_i\}) = [e_i], y(\{r_i\}) = [f_i] \ (i = 1, 2, 3) \). Then \( x \) and \( y \) are totally noncompatible but not complementary.

### 6.4 Sum logics and joint distributions of Urbanik type

The notion of a sum logic has its motivation in the quantum logic \( L(H) \) of closed subspaces of a (complex, separable) Hilbert space \( H \). By the spectral theorem, there is a one-to-one correspondence between the real observables on \( L(H) \) and the self-adjoint operators on \( H \). In particular, for every two bounded observables, their sum is defined as the observable corresponding to the operator sum of the corresponding bounded self-adjoint operators. In certain cases the sums of unbounded operators exist, too. We will try to find a generalization of this phenomenon in the study of "sum logics". We note that in the definition of a sum logic \( L \) we need not require \( L \) to be a lattice logic.

Let us recall the following notions (compare with Definition 44). A set \( S \) of states on \( L \) is said to be rich if the inclusion \( \{s \in S : s(a) = 1\} \subseteq \{s \in S : s(b) = 1\} \) implies \( a \leq b \). A set of states \( S \) on \( L \) is said to be order determining if \( s(a) \leq s(b) \) for all \( s \in S \) implies \( a \leq b \). For a set \( S \) of states, we denote by \( \text{conv}(S) \) the \( \sigma \)-convex envelope of \( S \).

Let \( L \) be a logic and let \( S \) be a rich \( \sigma \)-convex set of states on \( L \). For a real observable \( x \) on \( L \), set \( V(x) := \{s \in S : s(x^2) < \infty\} \). In view of Schwarz
inequality, \( s(x)^2 \leq s(x^2) \), and since \( \text{var}_s(x) = s(x^2) - s(x)^2 \), we have \( V(x) = \{ s \in S : \text{var}_s(x) < \infty \} \).

**PROPOSITION 101.** [Gudder, 1979] Let \( L \) be a logic and let \( S \) be a \( \sigma \)-convex set of states on \( L \) which is rich. Let \( x \) be a real observable on \( L \). Then \( x \) is bounded if and only if \( V(x) = S \).

**Proof.** If \( x \) is bounded, then \( x^2 \) is also bounded, hence \( V(x) = S \). Conversely, let \( x \) be unbounded. We will show that there is a state \( s \in S \) such that \( s(x^2) < \infty \) does not hold. Let \( r_n \in \omega(x) \) be numbers such that \( 2^{n+2} - 1 > |r_n| > 2^{n+1} \) \((n \in \mathbb{N})\). Let \( U_n \) be disjoint open balls in \( \mathbb{R} \) with diameter less than 1 and with centers at \( r_n \). Put \( a_n = x(U_n) \). Since any \( U_n \) is open and contains an element from the spectrum of \( x \), we have \( a_n \neq 0 \). Let \( s_n \in S \) be such that \( s_n(a_n) = 1 \). Since the elements \( a_i (i \in \mathbb{N}) \) are orthogonal, we obtain \( s_j(a_k) = 0 \) for \( j \neq k \). The state \( s = \sum_{j \in \mathbb{N}} 2^{-j} s_j \) belongs to \( S \). Assume that \( s(x^2) < \infty \). Then

\[
\infty > \int t^2 s(x(dt)) \geq \sum_{j \in \mathbb{N}} \int_{U_j} t^2 s(x(dt)) \geq \sum_{j \in \mathbb{N}} \frac{(2j+1 - \frac{1}{2})^2}{2^j} = \infty,
\]

which is absurd. \( \blacksquare \)

**DEFINITION 102.** We say that the real observables \( x_1, x_2, \ldots, x_n \) are summable (or that their sum exists) if

(i) the set \( V(x_1, x_2, \ldots, x_n) := \bigcup_{i \leq n} V(x_i) \) is order determining;

(ii) there is a unique real observable \( z \) such that \( V(z) \supseteq V(x_1, x_2, \ldots, x_n) \) and \( s(z) = s(x_1) + s(x_2) + \cdots + s(x_n) \) for all \( s \in V(x_1, x_2, \ldots, x_n) \).

The observable \( z \) is called the sum of the observables \( x_1, x_2, \ldots, x_n \), and we write \( z = x_1 + x_2 + \ldots + x_n \).

**DEFINITION 103.** A pair \((L, S)\), where \( L \) is a logic and \( S \) is a \( \sigma \)-convex rich set of states, is called a sum logic if the following two conditions are satisfied:

(i) every two bounded observables are summable;

(ii) every two compatible observables are summable.

The above definition implies that summability and the sum of observables \( x_1, x_2, \ldots, x_n \) are invariant with respect to permutations of the set \( \{1, 2, \ldots, n\} \) and with respect to the mappings \( x_i \to rx_i \) \((i \leq n)\) for every \( r \in \mathbb{R} \). Moreover, if \( x_1, x_2, \ldots, x_n \) are summable and if the sums \( y_1 = x_1 + x_2 + \cdots + x_k \) and \( y_2 = x_{k+1} + \cdots + x_n \) exist, then \( x_1 + x_2 + \cdots + x_n = y_1 + y_2 \). Notice also that if \( x_i, i \leq n \) are compatible and \( x_i = u_i f_i^{-1} \) \((i \leq n)\), then \( x_1 + \cdots + x_n = u (f_1 + \cdots + f_n)^{-1} \).

We will see that the logic \( L(H) \) with \( S = S(L(H)) \) is a sum logic, and so are the logics of projections in a von Neumann algebra. To our knowledge, a characterization of sum logics is not known.
Proposition 99 and Proposition 101 imply that every two bounded observables have a bounded sum. This allows us to prove the following statements.

PROPOSITION 104. Let \((L, S)\) be a sum logic. Then

(i) if \(x, y\) are bounded observables, then \(x = y\) if and only if \(s(x) = s(y)\) (that is, \(L\) has the Uniqueness property);

(ii) \(L\) is a lattice;

(iii) every state \(s \in S\) has the Jauch-Piron property (that is, for every \(s \in S\), if \(a, b \in L\) and \(s(a) = 1 = s(b)\), then \(s(a \wedge b) = 1\)).

Proof. (i) Suppose that \(s(x) = s(y)\) for all \(s \in S\). Let \(q_0\) be the proposition observable \(q_0(\{0\}) = 1\). Then the observables \(x + q_0\) and \(y + q_0\) are sums of the observables \(x\) and \(q_0\). Since the sums are unique, we have \(x + q_0 = y + q_0\). Hence \(x = y\).

(ii) Let \(a, b\) be elements of \(L\) and let \(q_a, q_b\) be the corresponding proposition observables. We will show that \((q_a + q_b)(\{2\}) = a \wedge b\). From \(s((q_a + q_b)(\{2\})) = 1\) it immediately follows that \(s(q_a + q_b) = 1\), hence \(s(a) = 1 = s(b)\). Since \(S\) is rich, it follows that \((q_a + q_b)(\{2\})\) is a lower bound of \(a, b\). Let \(c \in L\) be such that \(c \leq a, b\). Then the equality \(s(c) = 1\) implies \(s(a) = 1\) and \(s(b) = 1\). Hence, \(s(q_a + q_b) = s(a) + s(b) = 2\). Since 2 is the maximal point of the spectrum \(\omega(q_a + q_b)\), we see that \(s(q_a + q_b)(\{2\}) = 1\). This implies that \(c \leq (q_a + q_b)(\{2\})\).

We have shown that the infimum \(a \wedge b\) exists in \(L\) for every \(a, b \in L\), i.e., \(L\) is a lattice.

(iii) Suppose that \(s(a) = 1\) and \(s(b) = 1\) for \(s \in S\). By the proof of (ii), we conclude that \(s(a \wedge b) = 1\). Thus \(s\) has the Jauch-Piron property. \(\blacksquare\)

The notion of another type of joint distribution is based on the Cramér-Wold theorem in classical probability theory. Let \(p\) be a probability measure on \(B(\mathbb{R}^k)\). For \(a \in \mathbb{R}^k\), define \(h_a : \mathbb{R}^k \to \mathbb{R}\) by putting \(h_a(r) = a \cdot r = \sum_{i \leq k} a_i r_i\). If \(\alpha\) is a real number, then the set \(\{r : a \cdot r \leq \alpha\} = P_a(\alpha)\) is a closed half-space in \(\mathbb{R}^k\). Further, we have \(p(P_a(\alpha)) = p(h_a^{-1}(\infty, \alpha])\). We recall that the Cramér-Wold theorem states that if \(p_1\) and \(p_2\) are probability measures on \(B(\mathbb{R}^k)\) such that \(p_1(P_a(\alpha)) = p_2(P_a(\alpha))\) for every \(a \in \mathbb{R}^k\) and every \(\alpha \in \mathbb{R}\), then \(p_1 = p_2\).

DEFINITION 105. We say that the observables \(x_1, x_2, \ldots, x_n\) on a sum logic have a joint distribution of Urbanik type in a state \(s\) if the following two conditions are satisfied:

(i) the sum \(\sum_{i \leq n} a_i x_i\) exists for any \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\),

(ii) there exists a probability measure \(p\) on \(B(\mathbb{R}^n)\) such that, for any \(a \in \mathbb{R}^n\) and any \(\alpha \in \mathbb{R}\),

\[
p(p_a(\alpha)) = s((\sum_{i \leq n} a_i x_i)(\infty, \alpha])\]
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If such a measure \( p \) exists, we call it the joint probability distribution of Urbanik type for the observables \( x_1, x_2, \ldots, x_n \) in the state \( s \). We will write \( p = s_{x_1, \ldots, x_n} \).

By using results from classical measure theory the following statement was proved ([Urbanik, 1961], see also [Pták and Pulmannová, 1991, Th. 5.5.3]). Let us denote by \( \Phi^s_a \) the characteristic function of the observable \( \sum_{i \leq n} a_i x_i \) in the state \( s \), i.e. let us write

\[
\Phi^s_a(u) = \int_{\mathbb{R}} e^{iut} s\left(\sum_{i \leq n} a_i x_i(\cdot)\right)(dt).
\]

THEOREM 106. The observables \( x_1, x_2, \ldots, x_n \) have a joint distribution of Urbanik type in a state \( s \) if and only if the function \( \Phi(a) = \Phi^s_a(1) : \mathbb{R}^n \rightarrow \mathbb{C} \) is the characteristic function of some \( n \)-dimensional probability distribution.

PROPOSITION 107. Let \( x_1, x_2, \ldots, x_n \) be compatible observables on a sum logic \((L, S)\). Then the joint probability distribution of Urbanik type exists in every state \( s \in S \).

Proof. Let \( w \) be the joint observable of \( x_1, x_2, \ldots, x_n \). Then we have \( x_i = w \cdot \pi_i^{-1} \), where \( \pi_i : \mathbb{R}^n \rightarrow \mathbb{R} \) is the corresponding projection \((i \leq n)\). By the functional calculus for compatible observables we obtain

\[
\sum_{i \leq n} a_i x_i = w \cdot \left(\sum_{i \leq n} a_i \pi_i\right)^{-1}
\]

for any \( a \in \mathbb{R}^n \). Hence, for any \( E \in \mathcal{B}(\mathbb{R}) \) we have

\[
s\left(\sum_{i \leq n} a_i x_i\right)(E) = s \cdot w\left(\sum_{i \leq n} a_i \pi_i\right)^{-1}(E),
\]

where \( s \cdot w \) is the probability measure on \( \mathcal{B}(\mathbb{R}^n) \) which is identical with the joint distribution of Urbanik type for the observables \( x_1, x_2, \ldots, x_n \) in the state \( s \). Indeed, we have

\[
\left(\sum_{i \leq n} a_i \pi_i\right)^{-1}(-\infty, \alpha] = P_a^s(\alpha).
\]

It should be noted that the joint distribution of Urbanik type for compatible observables is identical with the joint distribution of Gudder type. As a consequence, we obtain the following result.

PROPOSITION 108. Let \((L, S)\) be a sum logic and let \( x_1, x_2, \ldots, x_n \) be bounded observables on \( L \). If \( x_1, x_2, \ldots, x_n \) have the joint distribution of Gudder type in a state \( s \in S \) then they also have the joint distribution of Urbanik type in the state \( s \), and both joint distributions are identical.
Proof. The proof follows by compatibility of the reduced observables \( x_i \land c (i \leq n) \), where \( c \) denotes the commutator of \( x_1, x_2, \ldots, x_n \).

The converse of Proposition 107 has been proved for discrete observables only [Gudder, 1966; Urbanik, 1985], [Pták and Pulmannová, 1991, Prop. 5.5.5]. The more general case needs special properties of sum logics, which are satisfied, e.g., by the Hilbert space logic.

7 THE LOGIC OF CLOSED SUBSPACES OF A HILBERT SPACE

In this section we consider the Hilbert space logic in more detail. This logic plays an important role in investigations into the mathematical foundations of quantum mechanics. We restrict ourselves to a complex separable Hilbert space \( H \), although many results remain valid in more general situations. (It should be noted that there is a way of extending several techniques of Hilbert space logics to Prehilbert spaces. Interesting results along this line can be found in [Chetcuti, 2005; Chetcuti and Dvurečenskij, 2003; Dvurečenskij, 1993; Hamhalter and Pták, 1987; Pták and Weber, 2001; Buhagiar and Chetcuti, 2005; Cattaneo and Marino, 1986; Mushtari and Matvejchuk, 1985], etc.)

By a subspace we here always mean a linear subspace of \( H \) and by a closed subspace we mean a subspace of \( H \) closed in the norm induced by the scalar product.

Let \( M, N \) be subspaces of \( H \). Write \( M + N = \{ x + y : x \in M, y \in N \} \), denote by \( \overline{M} \) the norm closure of \( M \) in \( H \), and set \( M' = \{ x \in H : \langle x, y \rangle = 0 \ \forall y \in M \} \). Obviously, \( M' \) is a closed subspace of \( H \). Consider the partial ordering of \( L(H) \) defined by set inclusion, and denote the least element by 0 (i.e., \( 0 = \{ 0 \} \)), and the greatest element by 1 (thus \( 1 = H \)). It is easily seen that with the inclusion ordering, \( L(H) \) becomes a lattice with the operations \( \land, \lor \) defined by \( M \land N = M \cap N \) and \( M \lor N = M + N \).

The following result is the so called projection theorem.

THEOREM 109. If \( M \in L(H) \), then every vector \( x \in H \) can be uniquely written in the form \( x = y + z \), where \( y \in M \) and \( z \in M' \).

Proof. (see, e.g., [Halmos, 1951]). We may assume that \( x \in H \setminus M \). Put \( \delta = \inf \{ ||x - y|| : y \in M \} \). By the definition of \( \delta \), there exists a sequence \( \{ y_n \} \) in \( M \) such that \( ||y_n - x|| \rightarrow \delta \). We then obtain

\[
||y_n - y_m||^2 = 2||y_n - x||^2 + 2||y_m - x||^2 - 4\|\frac{1}{2}(y_n + y_m) - x\|^2
\leq 2||y_n - x||^2 + 2||y_m - x||^2 - 4\delta^2 \rightarrow 2\delta^2 + 2\delta^2 - 4\delta^2 = 0.
\]

Thus \( \{ y_n \} \) is a Cauchy sequence. Since \( M \) is closed and \( H \) complete, there exists \( y \in M \) such that \( y_n \rightarrow y \). Since the norm on \( H \) is continuous, we obtain \( ||y - x|| = \lim_{n \rightarrow \infty} ||y_n - x|| = \delta \). We will show that \( z = x - y \in M' \). Suppose that for some
v ∈ M w have α = β(z, v), where β is a real number. Using the equality ||z|| = δ, we obtain

\[ 0 = \delta^2 - \delta^2 \leq ||z + \alpha v||^2 - ||z||^2 \leq 2\beta(z, v)^2 + \beta^2|z, v|^2||v||. \]

This implies that for certain small negative values of β we find \( \langle z, v \rangle = 0 \). We conclude that \( x = y + z \), where \( y \in M, z \in M' \). This expression is unique: if \( x = y_1 + z_1 \), then \( y - y_1 = z - z_1 \) and since \( y - y_1 \in M, z - z_1 \in M' \), we obtain \( y = y_1, z = z_1 \).

**THEOREM 110.** The set \( L(H) \) endowed with the inclusion ordering and with the orthocomplementation operation defined above is a lattice logic.

**Proof.** The infimum \( \land \) of two elements becomes set intersection. It follows that the infimum of an arbitrary family of elements of \( L(H) \) exists and becomes set intersection. Assume that \( M_n \in L(H) \) (\( n \in \mathbb{N} \)). Then \( \bigvee M_n = \bigwedge \{ M \in L(H) : M \supset \bigcup_n M_n \} \). Thus, \( L(H) \) is a σ-complete lattice. The mapping \( M \rightarrow M' \) is an orthocomplementation. Indeed, the inclusion \( M \subset N \) obviously implies \( N' \subset M' \). Further, \( M \land M' = 0 \), because \( x \in M \) and \( x \in M' \) yields \( \langle x, x \rangle = 0 \), which means \( x = 0 \). The inclusion \( M \subset M'' \) is obvious. Assume that \( x \in M'' \setminus M \). By Theorem 109, \( x = y + z \) with \( y \in M \) and \( z \in M' \). We have \( z = x - y \in M'' \), and therefore \( z \in M'' \land M' \). Thus \( z = 0 \), and hence \( M = M'' \). Finally, assume that \( M \subset K \) and \( M' \land K = 0 \). If \( x \in K \), then we have \( x = y + z \) for \( y \in M \) and \( z \in M' \). But \( z = x - y \in K \), and, therefore, \( z = 0 \). Thus, \( M = K \), and we have verified the orthomodular law.

By an operator on \( H \) we mean a linear mapping \( A : H \rightarrow H \). The norm of an operator \( A \) is defined by \( ||A|| = \sup\{||Ax|| : x \in H, ||x|| = 1\} \). An operator \( A \) is bounded if the norm of \( A \) is finite. If \( A \) is bounded, the adjoint operator \( A^* \) of \( A \) is defined by the equality \( \langle Ax, y \rangle = \langle x, A^*y \rangle \) for any \( x, y \in H \). A bounded operator \( A \) is self adjoint if \( A = A^* \). A projection operator (also called a projection or a projector) is a bounded operator \( P \) such that \( P = P^2 = P^* \).

**THEOREM 111.** To every closed subspace of \( H \) we can associate a unique projection, and every projection is associated with a unique closed subspace of \( H \).

**Proof.** Assume that \( N \in L(H) \). The projection associated with \( N \), denoted by \( P^N \), is defined as follows: \( P^N x = y \), where \( y \) is the unique vector of \( N \) given by the decomposition \( x = y + z \) in the sense of Theorem 109. On the other hand, if \( P \) is a projection, the set \( N_P = \{ x \in H : Px = x \} \) is the subspace of \( H \) associated with \( P \). One can simply verify that \( P^N \) is a projection, \( N_P \) is a closed subspace, and that \( N_P \) is \( N \) and \( P^N_P = P \).

The set of all projections on \( H \) can be endowed with a partial ordering by writing \( P \leq Q \) if \( PQ = QP = P \). The following statements are well known and easy to prove (e.g., [Halmos, 1951]).
PROPOSITION 112.

(i) Let $M, N$ be closed subspaces of $H$ and let $P^M, P^N$ be the corresponding projections. Then $M \subset N$ if and only if $P^M = P^M P^N = P^N P^M$, i.e., $P^M \leq P^N$.

(ii) $M \perp N$ if and only if $P^M P^N = P^N P^M = 0$ if and only if $P^M + P^N$ is a projection.

(iii) $M, N$ are compatible in $L(H)$ if and only if $P^M P^N = P^N P^M = 0$ if and only if $P^M + P^N$ is a projection.

(iv) For two projections $P, Q$, the operator $Q - P$ is a projection if and only if $P^M P^N = P^N P^M$.

(v) $PQ$ is a projection if and only if $PQ = QP$.

(vi) If $P$ is a projection, then $1 - P$ is also a projection, and $N_{1-P} = N_P$.

The states on the logic $L(H)$ are characterized by the famous Gleason theorem [Gleason, 1957] stated below.

THEOREM 113. (Gleason theorem) Let $H$ be a separable Hilbert space with $\dim H \geq 3$. If $s \in S(L(H))$ is a pure state on $L(H)$, then there exits a unit vector $v_s \in H$ such that $s(P) = \langle v_s, P v_s \rangle$ ($P \in L(H)$). Moreover, every state on $L(H)$ can be written as a $\sigma$-convex combination of pure states such that the unit vectors corresponding to them are mutually orthogonal.

Gleason's theorem implies that the pure states $s$ on $L(H)$ are in a one-to-one correspondence with unit vectors $v_s$ in $H$ (i.e., $v_s \in H, \|v_s\| = 1$) given by $s(P) = \langle P v_s, v_s \rangle$. Therefore pure states are also called vector states. Moreover all states on $L(H)$ are in a one-to-one correspondence with the positive self-adjoint trace-class operators with unit trace (so-called density operators). That is, to every state $s$ there corresponds a positive operator $T_s = \sum_{i \in \mathbb{N}} c_i P^{[v_i]}$, where $\sum_{i \in \mathbb{N}} c_i = 1$, and \{\(v_i : i \in \mathbb{N}\)\} is a complete orthonormal set of vectors in $H$. We then have that $s(P) = \text{tr}(T_s P) = \sum_{i \in \mathbb{N}} c_i \langle P v_i, v_i \rangle$.

Let us note that Gleason's theorem also holds for separable real and quaternionic Hilbert spaces with dimension at least three and it can be extended to certain nonseparable Hilbert spaces (see [Dvurečenskij, 1993; Hamhalter, 2003]).

Let $(\Omega, \Sigma)$ be a measurable space. A spectral measure is a mapping $E : \Sigma \rightarrow L(H)$ such that the following conditions are satisfied:

(i) $E(\Omega) = 1$;

(ii) if $S_1, S_2 \in \Sigma$ and $S_1 \cap S_2 = \emptyset$, then $E(S_1) \perp E(S_2)$;

(iii) if $S_i, i \in \mathbb{N}$ is a collection of pairwise disjoint sets of $\Sigma$, then $E(\bigcup_{i \in \mathbb{N}} S_i) = \sum_{i \in \mathbb{N}} E(S_i)$, where the sum on the right converges in the strong operator topology.
In other words, a spectral measure is an observable \( E : \Sigma \to L(H) \). If \( \Sigma = \mathcal{B}(\mathbb{R}) \), then \( E \) is a real observable. The real observables on \( L(H) \) are characterized by the spectral theorem. Before stating it, let us first generalize the notion of bounded operator. Let \( D \) be a dense subspace of \( H \) and let \( T : D \to H \) be a linear mapping. Call \( T \) again an operator on \( H \). Then one can define the operator \( T^* \) adjoint to \( T \) such that the following conditions are satisfied: The domain of \( T^* \) is determined by the requirement \( D(T^*) = \{ y \in H : \) there is a unique \( y^* \in H \) such that \( \langle Tx, y \rangle = \langle x, y^* \rangle \) for any \( x \in D(T) \} \), and we define \( T^*y = y^* \) \((y \in D(T^*))\). An operator \( T \) is called self-adjoint if \( T = T^* \).

**THEOREM 114 (Spectral theorem).** For any self-adjoint operator \( A \) on a Hilbert space \( H \) there exists a unique observable \( P^A : \mathcal{B}(\mathbb{R}) \to L(H) \) such that

(i) the domain \( D(A) \) of \( A \) consists of all vectors \( x \in H \) for which the integral \( \int \lambda^2 \langle P^A(d\lambda)x, x \rangle \) converges,

(ii) for any \( x \in D(A) \) and any \( y \in H \) we have \( \langle Ax, y \rangle = \int \lambda \langle P^A(d\lambda)x, y \rangle \).

Conversely, for any real observable \( E \) there exists a unique self-adjoint operator \( A \) such that \( E = P^A \).

For the proof of the spectral theorem see, e.g., [Halmos, 1951]. The spectral measure corresponding to a self-adjoint operator \( A \) is also sometimes called the spectral resolution of \( A \). The spectrum of an operator \( A \) is defined as the set of all complex numbers \( c \) such that the operator \( A - cI \) has no bounded inverse mapping. It can be proved that the spectrum of a self-adjoint operator \( A \) coincides with the spectrum of the corresponding observable \( P^A \).

Let \( s \) be a state on \( L(H) \) and let \( T \) be the corresponding density operator. Let \( A \) be a self-adjoint operator and let \( P^A \) be the corresponding observable. Then for the expectation of \( A \) (equivalently, of \( P^A \)) in the state \( s \) we have

\[
s(A) = s(P^A) = \int \lambda s(\langle P^A(d\lambda)x, x \rangle) = tr(TA),
\]

if the integral converges.

Let us conclude this section by showing that the Hilbert space logic is a sum logic in the sense of Definitions 102, 103. This fact is important in quantum axiomatics.

By the spectral theorem, if \( A \) is a self-adjoint operator, we can identify it with the corresponding observable, and hence view \( A \) as a mapping from \( \mathcal{B}(\mathbb{R}) \) to \( L(H) \). If \( f : \mathbb{R} \to \mathbb{R} \) is a measurable function, then \( f(A) = Af^{-1} \) is an observable too (for example, \( A^2 = Af^{-1} \), where \( f(t) = t^2 \)).

Denote by \( D(A) \) the domain of a self-adjoint operator \( A \). Recall that, in case \( A \) is unbounded, \( D(A) \) is a dense subset of \( H \). Moreover, we can write

\[
D(A) = \{ u \in H : \int t^2 \langle A(dt)u, u \rangle < \infty \} = \{ u \in H : s_u(A^2) < \infty \}.
\]
PROPOSITION 115. Let $S_v$ be the set of all vector states on $L(H)$. Then $S \subset S_v$ is rich for $L(H)$ if and only if $S = S_v$.

Proof. Richness of $S_v$ is obvious. If $S \neq S_v$, then there is a vector $v \in H$ such that $s_v \notin S$. Let $[v]$ be the subspace generated by the vector $v$. Then the equality $s_u([v]) = 1$ holds for no $s_u \in S$, contradicting the hypothesis that $S$ is rich. ■

COROLLARY 116. Let $A$ be a self-adjoint operator. Then the following statements are equivalent: (1) the set $S(A)$ of all vector states corresponding to the vectors of $D(A)$ is rich for $L(H)$; (2) $D(A) = H$; (3) $A$ is bounded.

PROPOSITION 117. For a linear subspace $K$ of $H$, write $S(K) = \{s_u : u \in K, \|u\| = 1\}$. Then $K$ is dense in $H$ if and only if $S(K)$ is order determining for $L(H)$.

Proof. Let $K$ be dense. Then if $s_u(P) \leq s_u(Q)$ for all unit vectors in $K$, we infer that $\langle Pu, v \rangle \leq \langle Qu, v \rangle$ for all $v \in H$, and hence $P \leq Q$. This shows that $S(K)$ is order determining.

Conversely, let $K_0$ be a subset of $H$ be such that the set $S(K_0) = \{s_u : u \in K\}$ is an order determining set of states. Let $K_1$ be the linear subspace generated by $K_0$. Obviously, $S(K_1)$ is order determining for $L(H)$. Let $K$ be the closure of $K_1$ in $H$. Then for every $u \in K_0$ we have $s_u(K) = 1$. Since $S(K_0)$ is order determining, we have $K = H$. ■

Observe that if $s = \sum_{i \in \mathbb{N}} a_i s_i$ and $s(A^2) < \infty$, then $s_i(A^2) < \infty$ for all $i \in \mathbb{N}$. Moreover, if $P, Q \in L(H)$ and if $s_i(P) \leq s_i(Q)$ for all $i$, then $s(P) \leq s(Q)$. This implies that the set $\{s \in S(L(H)) : s(A^2) < \infty\}$ is order determining for $L(H)$ if and only if the set of vector states $\{s_u : s_u(A^2) < \infty\}$ is order determining for $L(H)$.

If $A$ is a self-adjoint operator, its domain $D(A)$ is dense in $H$. If $A, B$ are self-adjoint operators, then their sum $A + B$ is defined on $D(A) \cap D(B)$. It is known that $A + B$ need not be self-adjoint. But if at least one of $A, B$ is bounded, then $A + B$ is self-adjoint (see e.g. [Dunford and Schwarz, 1957], [Prugovecki, 1971]). If $A, B$ commute, i.e. if they correspond to compatible observables, then we have by functional calculus of compatible observables that $A + B$ is self-adjoint.

Let us finally describe an algebraic construction in connection with sum logics. Let $x, y$ be two bounded real observables on a sum logic. The so called Segal product is defined by

$$x \circ y = \frac{1}{4}[(x + y)^2 - (x - y)^2].$$

In [Hudson and Pulmannová, 1993], the following statement was proved.

THEOREM 118. Let $L$ be a sum logic such that on the set of $O_b(L)$ of all bounded observables on $L$, the Segal product is distributive (with respect to the sum of observables). Then $O_b(L)$ admits the structure of a Jordan algebra.
8 JOINT DISTRIBUTIONS ON THE HILBERT SPACE LOGIC

The main result of this section is the statement that observables having a joint probability distribution of Gudder type in a state $s$ also have a joint probability distribution of Urbanik type in the state $s$, provided that all required sums exist.

Let $L(H)$ be the logic of a separable Hilbert space $H$ (real or complex) with $\dim H \geq 3$. Let $B(M)$ be a Borel $\sigma$-algebra of subsets of a separable Banach space $M$ (later on we will restrict ourselves to the real observables). Recall briefly our notations. If $C$ is a closed subspace of $H$, then $P^C$ denotes the corresponding projection. For a vector $v$, the symbol $[v]$ denotes the one-dimensional subspace generated by $v$. If $\|v\| = 1$, then $s_v : C \rightarrow \langle v, P^C v \rangle$ denotes the corresponding state on $L(H)$. By Gleason's theorem, for every state on $L(H)$ there is a density operator $T$ with $s(C) = s(P^C) = tr(TP^C)$ such that $T$ is positive, self-adjoint and has a unit trace.

In agreement with the preceding sections, we write, for all $d = (d_1, d_2, \ldots, d_n) \in D^n$, $D = \{-1, 1\}$, and all $F \in L(H)$, $F^{d_i} = F$ provided $d_i = 1$, and $F^{d_i} = F'$ provided $d_i = -1$.

THEOREM 119. Let $v$ be a unit vector in $H$. The observables $x_1, x_2, \ldots, x_n$ on $L(H)$ have a joint distribution of Gudder type in the state $s_v$ if and only if for any $E_1, E_2, \ldots, E_n \in B(M)$ and any permutation $p$ of the set $\{1, 2, \ldots, n\}$ the following equality holds:

\[(8) \quad p^{x_1(E_1)} \ldots p^{x_n(E_n)} v = p^{x_{p(1)}(E_{p(1)})} \ldots p^{x_{p(n)}(E_{p(n)})} v.\]

For the proof of this theorem we need some simple lemmas the proofs of which are left to the reader.

LEMMA 120. If the equality (8) holds for a vector $v \in H$, then

\[p^{x_1(E_1)} \wedge \ldots \wedge x_n(E_n) v = p^{x_1(E_1)} \ldots p^{x_n(E_n)} v.\]

LEMMA 121. Let $\{v_i : i \in \mathbb{N}\}$ be an orthonormal set of vectors. If $\|v\|^2 = \sum_{i \in \mathbb{N}} |\langle v, v_i \rangle|^2$ for a vector $v \in H$, then $v = \sum_{i \in \mathbb{N}} \langle v, v_i \rangle v_i$.

LEMMA 122. Let $V_1, V_2, \ldots, V_n \in L(H)$ and let $v \in H$, $v \neq 0$. Let $v \in \bigwedge_{i \leq n} V_i^{d_i}$ for a $d \in D^n$. Then

\[P^{V_{p(1)}} \ldots P^{V_{p(n)}} v = P^{V_1} \ldots P^{V_n} v\]

for every permutation $p$ of the set $\{1, 2, \ldots, n\}$.

Let us now prove Theorem 119.

Proof. Assume that the equality (8) holds for $v \in H$, $\|v\| = 1$. Then by Lemma 120 we have

\[\sum_{d \in D^n} s_v(\bigwedge_{i \leq n} x_i(E_i)^{d_i}) = \sum_{d \in D^n} \langle v, P^{\bigwedge_{i \leq n} x_i(E_i)^{d_i}} v \rangle \]

\[= \sum_{d \in D^n} \langle v, p^{x_1(E_1)}^{d_1} \ldots p^{x_n(E_n)}^{d_n} v \rangle = \langle v, v \rangle = 1.\]
Using Theorem 87, we get that the observables \( x_1, x_2, \ldots, x_n \) have a joint distribution of Gudder type in the state \( s_v \).

Suppose that a joint distributions of Gudder type for the observables \( x_1, x_2, \ldots, x_n \) in the state \( s_v \) exists. Let \( E_1, E_2, \ldots, E_n \in \mathcal{B}(M) \). By Theorem 87, we have

\[
\sum_{d \in D^n} \langle v, P^{\wedge_{i \leq n} x_i(E_i)^d} v \rangle = 1.
\]

This equality yields

\[
1 = ||v||^2 = \sum_{d \in D^n} ||\langle v, \frac{P^M(d)}{||P^M(d)v||} v \rangle||^2,
\]

where we put \( M(d) = \wedge_{i \leq n} x_i(E_i)^d \) (\( d \in D^n \)). Obviously, the vectors \( \{P^M(d)v : d \in D^n\} \) are mutually orthogonal. Applying Lemma 121, we see that the vector \( v \) is a linear combination of the vectors \( P^M(d)v \). By Lemma 122, condition (8) is satisfied. This completes the proof of Theorem 119.

The following theorem gives a necessary and sufficient condition for the existence of a joint distribution of Gudder type.

**THEOREM 123.** Let \( x_1, x_2, \ldots, x_n \) be observables and let \( s \) be a state with the density operator \( T = \sum_{i \in N} c_i P^{[v_i]} \).

(i) The joint distribution of Gudder type for \( x_1, x_2, \ldots, x_n \) in the state \( s \) exists if and only if the condition 8 of Theorem 119 is satisfied for each vector \( v_i (i \in N) \).

(ii) If \( x_1, x_2, \ldots, x_n \) are real bounded observables and if \( A_1, A_2, \ldots, A_n \) are the corresponding bounded self-adjoint operators on \( H \), then the following conditions are equivalent:

(a) the observales \( x_1, x_2, \ldots, x_n \) have a joint distribution of Gudder type in the state \( s \);

(b) the condition (8) of Theorem 119 holds for each vector \( v_i (i \in N) \);

(c) the equality \( A_1 \ldots A_n v_i = A_{p(1)} \ldots A_{p(n)} v_i \) \( (i \in N) \) holds for any permutation \( p \) of the set \( \{1, 2, \ldots, n\} \);

(d) the equality \( A_1 \ldots A_n T = A_{p(1)} \ldots A_{p(n)} T \) holds for every permutation \( p \) of the set \( \{1, 2, \ldots, n\} \).
Proof. By Theorem 87 we see that a joint distribution of Gunder type in the state \( s = \sum_{i \in \mathbb{N}} s_i \) exists iff it exists in state \( s_i \) for all \( i \in \mathbb{N} \). Combining this with Theorem 119, we obtain that (i) holds. From the properties of spectral measures and Theorem 119 we deduce that statement (ii) holds.

THEOREM 124. Let \( \{x_\alpha\}_{\alpha \in I} \) be a system of observables on \( L(H) \). Denote by \( H_0 \) the set of all vectors in \( H \) that satisfy the condition (8) for any finite choice \( x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_n}, \alpha_1, \alpha_2, \ldots, \alpha_n \in I \). Then \( H_0 \) is a closed linear subspace of \( H \) and the equality \( H_0 = \text{com}\{x_\alpha : \alpha \in I\} \) holds. Moreover, if \( s \) is a state with the density operator \( T = \sum_{i \in \mathbb{N}} c_i P^{[v_i]} \), then the joint distribution of Gunder type for the observables \( \{x_\alpha : \alpha \in I\} \) in the state \( s \) exists if and only if \( v_i \in H_0 \) for all \( i \in \mathbb{N} \).

Proof. It is easy to check that \( H_0 \) is a closed linear subspace of \( H \). Since \( L(H) \) is a complete lattice, the commutator \( C := \text{com}\{x_\alpha : \alpha \in I\} \) exists. Moreover, since \( L(H) \) is separable (in the sense that every orthogonal set of projections is at most countable), we can find a countable subset \( J \) of \( I \) such that \( C = \text{com}\{x_j : j \in J\} \) (see Proposition 22). Let \( v \) be a unit vector and \( s_v \) the corresponding state. By Theorem 119, \( v \in H_0 \) if and only if the joint distribution of Gunder type for \( \{x_\alpha : \alpha \in I\} \) exists in the state \( s_v \). Since \( C = \text{com}\{x_j : j \in J\} \), we deduce (Theorem 87) that the joint distribution for the observables \( \{x_\alpha : \alpha \in I\} \) in the state \( s_v \) exists if and only if \( s_v(C) = 1 \), i.e., if and only if \( v \in C \). It follows that \( H_0 = C \). The remaining part of the proof follows by Theorem 123.

The next theorem is a converse of Proposition 107 for bounded observables on \( L(H) \). We will follow the proof in [Varadarajan, 1985, Theorem 4.25]. Let us first introduce some notations. Let \( x_j (j \leq k) \) be bounded observables on \( L(H) \), and let \( A_j \) be the bounded self-adjoint operator corresponding to \( x_j \). If \( c_1, c_2, \ldots, c_n \) are real numbers, put \( c = (c_1, \ldots, c_k), A(c) = c_1A_1 + c_2A_2 + \cdots + c_nA_n \). Let \( E \mapsto P^{A(c)}(E) := Q(c; E) \) be the spectral measure of \( A(c) \). If \( \phi \in H \) is a unit vector, we have \( s_\phi(P^{A(c)}(E)) = s_\phi(c; E) = \|Q(c; E)\| \), so that \( s_\phi(c; E) \) is a probability measure on \( B(\mathbb{R}) \). Let us recall that observables \( x_1, \ldots, x_k \) have a joint distribution of Urbanik type in a state \( s_\phi \) if there is a probability measure \( p_\phi \) on \( B(\mathbb{R}) \) such that \( p_\phi(E(c)) = s_\phi(c; E) \)

where \( E(c) = \{(t_1, t_2, \ldots, t_k) : c_1 t_1 + \cdots + c_k t_k \in E \} \).

For the proof of the theorem announced we need the following lemma.

LEMMA 125. Let \( B \) be a nonnegative self-adjoint operator and \( S, T \) two projections such that \( B \leq S \) and \( B \leq T \). Then \( B \leq S \wedge T \).
\textbf{Proof.} Let \(N_S, N_T\) be the ranges of \(S, T\), respectively. Then \(\|B^{1/2}v\|^2 \leq \|Sv\|^2\), and hence \(B\) vanishes on \(N_S^\prime\). Similarly, \(B\) vanishes on \(N_T^\prime\) and hence \(B = 0\) on \((N_S \cap N_T)^\prime\). Since \(B\) is self-adjoint, it leaves \(N_S \cap N_T\) invariant, and for \(v \in N_S \cap N_T\), \(\langle Bv, v \rangle \leq \|v\|^2\). Thus \(B \leq P_{N_S \cap N_T} = S \wedge T\). \(\square\)

**THEOREM 126.** The bounded observables \(x_1, x_2, \ldots, x_k\) on \(L(H)\) have a joint distribution of Urbanik type in all states on \(L(H)\) if and only if the operators \(A_1, A_2, \ldots, A_k\) corresponding to \(x_1, x_2, \ldots, x_k\) commute with one another.

**Proof.** Sufficiency follows by Proposition 107. To prove necessity, assume that the observables in question have a joint distribution of Urbanik type in all states on \(L(H)\). Let us restrict ourselves to the case \(k = 2\) and prove that \(A_1 A_2 = A_2 A_1\), the in the general case the argument is similar. For any nonzero \(\phi \in H\), we write \(\psi = (||\phi||)^{-1} \phi\) and define \(q_{\phi} := ||\phi||^2 p_\psi\). Then \(q_{\phi}\) is a nonnegative measure on \(B(\mathbb{R}^2)\) and \(q_{\phi}(\mathbb{R}^2) = ||\phi||^2\). Set \(q_0 = 0\). For any pair for vectors \(\phi, \phi' \in H\), let us define a \(\mathbb{C}\)-valued measure \(q_{\phi, \phi'}\) by the polarization formula

\[(q_{\phi, \phi'}) := \frac{1}{4}(q_{\phi+\phi'} - q_{\phi-\phi'} + iq_{-i\phi+\phi'} - iq_{-i\phi-\phi'}).\]

We claim that for each \(M \in B(\mathbb{R}^2)\), the map \((\phi, \phi') \mapsto q_{\phi, \phi'}(M)\) is bounded and sesquilinear on \(H \times H\). Indeed, let us write

\[\nu(M) := q_{\phi+\phi'}(M) - q_{\phi, \phi'}(M) - q_{\phi, \phi'}(M).\]

If \(M = E(c)\), \(E \in B(\mathbb{R})\), then \(q_{\phi}(M) = (Q(c; E)\phi, \phi)\) and hence

\[q_{\phi, \phi'}(E(c) = (Q(c; E)\phi, \phi'),\]

which entails \(\nu(M) = 0\). Since this equality holds for all \(M = E(c)\), it follows \(\nu = 0\). Similar arguments complete the proof that \((\phi, \phi') \mapsto q_{\phi, \phi'}\) is sesquilinear. From \(0 \leq q_{\phi, \phi'}(M) \leq ||\phi||^2\) we conclude that \((\phi, \phi') \mapsto q_{\phi, \phi'}\) is bounded. Using well-known techniques, we can construct a unique bounded self-adjoint operator \(P_M\) such that

\[q_{\phi, \phi'}(M) = (P_M \phi, \phi).\]

Clearly, \(0 \leq P_M \leq 1\) for all \(M\), \(P_0 = 0\), \(P_{\mathbb{R}^2} = 1\). Moreover, as \(M \mapsto q_{\phi, \phi'}(M)\) is a measure, \(P_M\) is countably additive with respect to the disjoint sets \(M_1, M_2, \ldots\). We also have \(q_{\phi, \phi'}(M) \geq 0\). Moreover, \(M_1 \subset M_2\) implies \(P_{M_1} \leq P_{M_2}\). If \(M = E(c)\), then \(P_M = Q(c; E)\).

We have to show that for any two sets \(E_1, E_2 \in B(\mathbb{R})\), the operators \(Q_1 := Q((1, 0); E_1)\) and \(Q_2 := Q((0, 1); E_2)\) commute. Since \(E_1 \times E_2 \subset E_1 \times \mathbb{R}\), we have

\[P_{E_1 \times E_2} \leq P_{E_1 \times \mathbb{R}} = Q_1.\]

Similarly,

\[P_{E_1 \times E_2} \leq P_{\mathbb{R} \times E_2} = Q_2.\]
From Lemma 125 we conclude that

\[ P_{E_1 \times E_2} \leq Q_1 \land Q_2. \]

Replacing \( E_2 \) by \( \mathbb{R} \setminus E_2 \), we obtain

\[ P_{E_1 \times \mathbb{R} \setminus E_2} \leq Q_1 \land (1 - Q_2). \]

The last two inequalities give us

\[ Q_1 \leq Q_1 \land Q_2 + Q_1 \land (1 - Q_2) \leq Q_1, \]

where the last inequality holds because \( Q_1 \land Q_2 \) and \( Q_1 \land (1 - Q_2) \) are orthogonal projections. Hence we obtain

\[ Q_1 = Q_1 \land Q_2 + Q_1 \land (1 - Q_2), \]

from which we conclude that \( Q_1 \) and \( Q_2 \) commute. \( \square \)

We now establish an important connection between the joint probability distributions of Gudder and Urbanik type on \( L(H) \) (\( L(H) \) can be regarded as a prototype of a sum logic). This extends the result of Proposition 108 to certain unbounded real observables on \( L(H) \). We do not go here into details concerning sums of unbounded operators, see e.g. [Pták and Pulmannová, 1991, Theorem 5.6.8].

**Theorem 127.** Let \( x \) and \( y \) be real observables on \( L(H) \), and let for all \( \alpha, \beta \in \mathbb{R} \) the observables \( \alpha x + \beta y \) exist. Then if the observables \( x \) and \( y \) have a joint distribution of Gudder type in a state \( s \), they also have a joint distribution of Urbanik type in the state \( s \), and these joint distributions coincide.

**Proof.** We sketch the proof of Theorem 127. In view of Theorem 123, it is sufficient to consider only vector states. Let \( v \) be a unit vector in \( H \) and let the joint distribution of Gudder type exist in the state \( s_v \). Applying Theorem 124, we find that \( v \in H_0 \), where \( H_0 = \text{com}(x, y) \). Let \( A_x \) and \( A_y \) denote the self-adjoint operators on \( H \) corresponding to \( x \) and \( y \), respectively. Then \( P_{H_0} \) commutes with \( A_x \) and \( A_y \). That is, \( H_0 \) reduces \( A_x \) and \( A_y \), the reduced operators \( A_x/H_0 \) and \( A_y/H_0 \) are self-adjoint and the corresponding observables \( x_0 \) and \( y_0 \) on \( L(H_0) \) are compatible. The observables \( \alpha x \) and \( \beta y \) are summable if and only if \( \alpha A_x + \beta A_y \) is essentially self-adjoint (i.e., \( \alpha A_x + \beta A_y \) has a unique self-adjoint extension \( \alpha A_x + \beta A_y \), corresponding to the observable \( \alpha x + \beta y \)). Then \( H_0 \) reduces also the self-adjoint operator corresponding to \( \alpha x + \beta y \), and \( (\alpha x + \beta y)_0 = \alpha x_0 + \beta y_0 \). Since \( x_0 \) and \( y_0 \) are compatible, they have a joint distribution of Urbanik type in the state \( s_v \) (restricted to \( H_0 \)). This joint distribution is identical with the joint distribution of Gudder type (Proposition 107). This means that there is a measure \( \mu \) on \( B(\mathbb{R}^2) \) such that

\[ \mu\{(r_1, r_2) : \alpha r_1 + \beta r_2 \in E\} = s_v(\alpha x_0 + \beta y_0)(E) \]
for all $\alpha, \beta \in \mathbb{R}$ and all $E \in \mathcal{B}(\mathbb{R})$. We can further write

$$s_v(\alpha x_0 + \beta y_0)(E)) = \langle v, P^{(\alpha x_0 + \beta y_0)(E)} \rangle_v = \langle v, P^{(\alpha x + \beta y)(E)} \rangle_v$$

This implies that the joint distribution of Urbanik type for $x$ and $y$ in the state $s_v$ exists. If $E, F \in \mathcal{B}(\mathbb{R})$, then

$$\mu(E \times F) = s_v(x_0(E) \wedge y_0(F)) = \langle v, P^{(E)}x_0(E) \rangle_v$$

which implies that the joint distribution of Gudder and Urbanik type for the observables $x$ and $y$ are equal (since they are equal for the observables $x_0$ and $y_0$).

The proof is complete.

The following example shows that the converse of Theorem 127 does not hold. Let us consider operators $P$ and $Q$ of momentum and position on $L^2(\mathbb{R})$. Thus the operator $Q$ has the domain $D(Q) = \{ f \in L^2(\mathbb{R}) : \int |r f(r)|^2 dr < \infty \}$, and $(Qf)(r) = rf(r)$ for $f \in D(Q)$. The operator $P$ has the domain $D(P) = \{ f \in L^2(\mathbb{R}) : \int |p \hat{f}(p)|^2 dp < \infty \}$, where $\hat{f}$ is the Fourier-Plancherel transform of $f(r)$, and $(Pf)(r) = -ih\frac{\partial f(r)}{\partial r}$. It is well-known that $P$ and $Q$ satisfy the Heisenberg uncertainty relation, and therefore they do not have a joint distribution of Gudder type in any state. In fact, they are even complementary. But it was proved in [Urbanik, 1961] that they have a joint distribution of Urbanik type in the state

$$\phi_1(r) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{r^2}{2}\right),$$

and do not have one in the state

$$\pi_2(r) = \frac{2^{1/2}r}{\pi^{1/4}} \exp\left(-\frac{r^2}{2}\right).$$

9 APPENDIX 1: UNCERTAINTY RELATIONS

Let $L$ be a quantum logic and $S(L)$ the state space of $L$. Let $x$ be a real observable on $L$. Let $\text{var}_s(x)$ denote the variance of $x$ in the state $s$. Put

$$V(x) = \{ s \in S(L) : \text{var}_s(x) < \infty \}.$$

For any two real observables, the following alternative possibilities can occur:

(A) $(\forall \varepsilon > 0)(\exists s \in V(x) \cap V(y))(\text{var}_s(x), \text{var}_s(y) < \varepsilon)$.

(B) $(\exists \varepsilon > 0)(\forall s \in V(x) \cap V(y))(\text{var}_s(x), \text{var}_s(y) \geq \varepsilon)$. 
If (B) occurs, we say that the uncertainty relation holds for $x$ and $y$.

For $t \in \mathbb{R}$, $\delta > 0$, put $U(t, \delta) = \{ r \in \mathbb{R} : |t - r| < \delta \}$. Notice that if $t$ belongs to the spectrum of $x$, then $x(U(t, \delta)) \neq 0$.

Recall that $L$ belongs to the class $\mathcal{L}_3$ if $S(L)$ is unital (i.e., to every $a \in L$ there is a state $s \in S(L)$ with $s(a) = 1$) and for any $a, b \in L$ such that $a \not\rightarrow b$ there is $s \in S(L)$ such that $s(a) = s(b) = 1$.

Let $x, y$ be real observables on $L$ with the spectra $\sigma(x), \sigma(y)$, respectively. The following two cases can occur:

(a) $(\forall(u, v, \delta) : u \in \sigma(x), v \in \sigma(y), \delta > 0) (\exists \eta_0 > 0) (\forall \eta, 0 < \eta < \eta_0) (x(U(u, \delta)) \leftrightarrow y(U(v, \eta)))$.

(b) $(\exists(u, v, \delta) : u \in \sigma(x), v \in \sigma(y), \delta > 0) (\forall \eta_0 > 0) (\exists \eta, 0 < \eta < \eta_0) (x(U(u, \delta)) \not\rightarrow y(U(v, \eta)))$.

In [Pulmannová, 1988], the following results were obtained.

**THEOREM 128.** Let $L$ be a logic with the unital state space. If for the observables $x$ and $y$ on $L$ the condition (a) holds, then the uncertainty relation does not hold. In other words, (a) $\implies$ (A).

Moreover, if $L$ is a lattice, then (a) implies that $x$ and $y$ are compatible.

**THEOREM 129.** Let $L \in \mathcal{L}_3$. Then the uncertainty relation does not hold for any pair of observables $x, y$.

**Proof.** Let $x, y$ be observables on $L$. By Theorem 128, (a)$\implies$(A). Suppose that (b) holds for $x$ and $y$. Then there are $u \in \sigma(x), \delta > 0, v \in \sigma(y)$ such that to any $\eta_0 > 0$ there is $\eta < \eta_0$ with $x(U(u, \delta)) \not\rightarrow y(U(v, \eta))$. As $L$ belongs to $\mathcal{L}_3$, there is a state $s \in S(L)$ such that $s(x(U(u, \delta))) = 1 = s(y(U(v, \eta)))$. Choosing $\eta$ sufficiently small, we obtain $\text{var}_s(x).\text{var}_s(y) < \epsilon$ for any given $\epsilon > 0$. \hfill \Box

10 APPENDIX 2: BELL INEQUALITIES ON QUANTUM LOGICS

J.S. Bell [Bell, 1960] introduced examples of inequalities valid in classical probability theory. However, these inequalities were found to be violated in certain quantum mechanical experiments. The Bell inequalities are sometimes considered as "tests" of "hidden variables", i.e., tests of the nearness of the probability model used to the classic Kolmogorov model. Many results can be found in the literature that suggest the non-existence of hidden variables in quantum mechanics (see e.g., [Kochen and Specker, 1967; Beltrametti and Cassinelli, 1981; Giuntini, 1991; Bell, 1960]). In the logico-algebraic approach to quantum mechanics, the standard model of the set of experimentally verifiable propositions of a physical system, which is based on a Boolean $\sigma$-algebra, is replaced by a more general model based on a "quantum logic". The quantum logic is commonly supposed to be an orthomodular $\sigma$-poset. In the standard Hilbert space approach, the quantum logic is the projection lattice of a Hilbert space.
There are several formulations of the problem of hidden variables in the logico-algebraic approach. Typically, the existence of hidden variables is related to the existence of a rich supply of dispersion-free states on the set of propositions. A physical system is considered to be “classical” if its set of propositions forms a Boolean \( \sigma \)-algebra. The problem that arises now is that there are Boolean \( \sigma \)-algebras which possess many states but no dispersion-free states. As a result, even certain classical systems would on this view not admit hidden variables. So the existence of hidden variables as a test of whether or not a system is “classical”, resp. “quantum” may not be entirely convincing. An attempt to overcome this difficulty has been made in [Deliyannis, 1971], where in the criterion of the existence of so-called quasi hidden variables the dispersion-free states have been replaced by dispersion-free quasistates, i.e., finitely additive states. Recently [Zisis, 2000], a suitable definition of so called approximate hidden variables has been found in terms of countably additive states. For our purpose, the approach using quasi hidden variables in terms of finitely additive states is more appropriate in order to find connections with certain kinds of Bell inequalities.

In this section, we will make the connection between the existence of quasi hidden variables and the Bell inequalities on orthomodular lattices ([Pulmannová, 2002]). We will consider Bell inequalities in a probabilistic formulation (see [Pitowski, 1989]). These Bell inequalities on quantum logics have been studied, e.g., in [Beltrametti and Mandryńska, 1991; Pykač and Santos, 1991; Pulmannová and Majerník, 1992; Pulmannová, 1994a; D’Andrea and Pulmannová, 1994; Dvurečenskij and Länger, 1995].

11 QUANTUM LOGICS AND PHYSICAL SYSTEMS

Again, by a quantum logic we mean a \( \sigma \)-complete orthomodular poset \( L \). Recall that a mapping \( h : L_1 \rightarrow L_2 \), where \( L_1 \) and \( L_2 \) are quantum logics is a \( \sigma \)-homomorphism (of quantum logics) if (1) \( h(1) = 1 \), (2) \( h(a') = h(a)' \), and (3) if \( a = \vee a_i \) for a sequence \( (a_i) \) of pairwise orthogonal elements of \( L_1 \), then \( h(a) = \bigvee h(a_i) \). A \( \sigma \)-homomorphism which in addition preserves all existing infima and suprema, is called a lattice \( \sigma \)-homomorphism. A surjective \( \sigma \)-homomorphism is an isomorphism if \( a \leq b \) iff \( h(a) \leq h(b) \).

If \( L \) is a logic, by a state on \( L \) we mean a countably additive state. By relaxing this condition to finite additivity, we speak on finitely additive states. The set of all states (the state-space of \( L \)) will be denoted by \( S(L) \).

Let \( L \) be a logic and \( s \) a state. For any element \( a \in L \), the dispersion of \( s \) at \( a \) is the number \( s(a) - s(a)^2 \). The total dispersion of \( s \) may be defined as

\[
\sigma(s) = \sup_{a \in L} \{ s(a) - s(a)^2 \}.
\]

A state \( s \) is dispersion-free if \( \sigma(s) = 0 \), i.e., if the range of \( s \) is \( \{0, 1\} \). Obviously, the latter notions make sense also for finitely additive states and we shall occasionally use them for this setup.
Let $L$ be a logic, and $\Omega, S$ two sets of \textit{finitely additive states}. We say that $\Omega$ \textit{generates} $S$ if there is a $\sigma$-algebra $\Sigma$ on $\Omega$ such that

(i) for every $a \in L$, the map $\omega \mapsto \omega(a)$ is $\Sigma$-measurable,

(ii) for every $s \in S$, there exists a probability measure $\mu_s$ on $(\Omega, \Sigma)$ such that for all $a \in L$,

$$s(a) = \int_{\Omega} \omega(a) d\mu_s(\omega).$$

In accord with Chapter 1, let $L$ be a logic and $S$ a set of (finitely additive, in general) states on $L$. $S$ is called \textit{order-determining} if the inequality $s(a) \leq s(b)$ for all $s \in S$ implies that $a \leq b$, and $S$ is called \textit{rich} if $a \not\leq b$ implies that there is $s \in S$ with $s(a) = 1, s(b) < 1$.

A \textit{system} is a pair $(L, S)$ where $L$ is a logic and $S$ is an order-determining set of states. If $(L, S)$ is a system, then a \textit{pure state} is an element of $S$ which cannot be written as a nontrivial convex combination of elements of $S$.

If $(L, S)$ and $(L', S')$ are systems, a mapping $f : L \to L'$ is a \textit{system imbedding} if $f$ is a $\sigma$-homomorphism of logics and

(1) for every $s \in S$, there exists $s' \in S'$ such that $s' \circ f = s$.

Observe that if $f$ is an imbedding, then $a \leq b$ iff $f(a) \leq f(b)$. By definition, a system $(L, S)$ is a \textit{subsystem} of $(L', S')$ if there is a system imbedding of $(L, S)$ in $(L', S')$.

Let us consider the following examples [Zisis, 2000].

\textbf{Example 1.} Let $L$ be a Boolean $\sigma$-algebra. Let $L$ possess an order-determining set of states. Consider the system $(L, S(L))$ associated with $L$. It can be derived from the Loomis-Sikorski theorem that there is a measurable space $(\Omega, \Sigma)$ and a set $M$ of finite measures on $(\Omega, \Sigma)$ such that $L$ is isomorphic to the quotient of $\Sigma$ by the $\sigma$-ideal $\{ A \in \Sigma : \mu(A) = 0 \text{ for all } \mu \in M \}$. This example has the following important special cases.

(1a). Let $(\Omega, \Sigma)$ be a measurable space. The system associated with it is the pair $(\Sigma, S(\Sigma))$.

(1b). Let $(\Omega, \Sigma, \nu)$ be a $\sigma$-finite measure space. The system associated with it is the pair $(L, S(L))$, where $L$ is the quotient of $\Sigma$ by the $\sigma$-ideal of sets of measure zero. Then $S(L)$ can be identified with the set of the probability measures on $\Sigma$ that are absolutely continuous with respect to $\nu$.

\textbf{Example 2.} Let $H$ be a separable Hilbert space, $\dim H \geq 3$. The system associated with it is the pair $(L, S)$, where $L$ is the logic of all closed subspaces of $H$ (equivalently, of all orthogonal projections) and $S$ is the set of all states on $L$.

Examples (1a) and (1b) are related to classical mechanics. Example (2) corresponds to standard quantum mechanics.

According to [Varadarajan, 1985, p.116-117], if $\omega$ is a dispersion-free state on a countably generated Boolean $\sigma$-algebra $L$, then there is an atom $p$ of $L$ with
\[
\omega(p) = 1. \text{ So if } L \text{ has no atoms, it has no dispersion-free states. Consequently, if } L \text{ is a logic with a countably generated Boolean } \sigma\text{-subalgebra with no atom, then } L \text{ has no dispersion-free states. It can is easily seen that a logic has a countably generated Boolean } \sigma\text{-subalgebra with no atoms iff } L \text{ admits an observable with the discrete spectrum empty. So we may conclude that while in Example (1a) all real-valued observables have nonempty discrete spectra, in Example (1b) there may be observables with empty discrete spectrum, like, e.g., the position observable.}

In standard quantum systems, if \( H \) is finite-dimensional, then all observables are discrete, while if \( \dim H = \infty \), there are observables, like position and momentum observables, with continuous spectra.

So the non-existence of dispersion-free states is not a criterion of whether a system is classical or quantum.

There is an important difference between examples (1b) and (2). While in the first case all pure states are dispersion-free, in the second case there are pure states which are not dispersion-free. In some hidden-variable approaches, it is required that only that the pure states be generated by dispersion-free states. But this criterion cannot be applied to quantum mechanical systems with no pure states.

The usual definition of a system that admits hidden variables is as follows.

**DEFINITION 130.** A system \((L, S)\) is deterministic if the set of dispersion free elements of \( S \) generates the entire \( S \).

**DEFINITION 131.** A system \((L, S)\) admits hidden variables if there is a set \( S_{HV} \) such that \( S_{HV} \supset S \) and \((L, S_{HV})\) is a deterministic system.

Recall that a logic \( L \) is concrete if \( L \) can be represented by a collection of subsets of a set [Pták and Pulmannová, 1991]. By Theorem 48, a logic \( L \) is concrete if and only if it admits an order-determining set of dispersion free states. Observe that for dispersion free states, the notions of rich and order-determining system of states coincide.

Recall further that the Jauch-Piron condition is satisfied for \( a, b \) if for every state \( s \) on \( L \), if \( s(a) = s(b) = 1 \), then \( s(a \land b) = 1 \).

**THEOREM 132.** [Deliyannis, 1971] Let \((L, S)\) be a system, \( \Omega \) a generating set of states for \( S \). Then \( L \) is isomorphic (in the quantum logic sense) to a logic which is represented by functions on the set \( \Omega \). If the states in \( \Omega \) are dispersion free, these functions are characteristic functions of subsets of \( \Omega \). Moreover, a necessary and sufficient condition for the above isomorphism to preserve infimum \( a \land b \) (or supremum \( a \lor b \)) is the Jauch-Piron condition for this particular pair.

**Remark.** The preceding theorem implies that if a system \((L, S)\) admits hidden variables, then \( L \) is a concrete logic. The converse need not be true. Indeed, a necessary and sufficient condition that a system \((L, S)\), where \( L \) is a concrete logic admits hidden variables is that all states in \( S \) can be extended to states on a Boolean \( \sigma \)-algebra of sets.

Example (1a) is clearly deterministic. But in Example (1b), if \( \nu \) is nonatomic, then every probability measure that is absolutely continuous with respect to \( \nu \) is
also nonatomic, so it cannot be dispersion free. Thus \( L \) has no dispersion free states, hence \((L, S(L))\) does not admit hidden variables.

In Example (2), the associate system \((L, S)\) is deterministic iff \( \dim H = 1 \). If \( \dim H \geq 3 \), \( H \) has no dispersion free states [Alda, 1981]. So in this case, \((L, S)\) does not admit hidden variables.

We see that the above definition does not distinguish classical and quantum systems. An interesting way to overcome this difficulty has been suggested in [Deliyannis, 1971].

**DEFINITION 133.** A system \((L, S)\) admits quasi hidden variables if the set of finitely additive dispersion free states of \( L \) generates \( S \).

The next lemma shows that a system \((B, S)\) associated with a Boolean \( \sigma \)-algebra \( B \) admits quasi hidden variables (see also [Deliyannis, 1971, Theorem 6]).

**LEMMA 134.** Let \( B \) be a Boolean \( \sigma \)-algebra, and let \( s \) be a (\( \sigma \)-additive) state on \( B \). Then \( s \) is generated by finitely additive dispersion free states on \( B \).

**Proof.** Let \( \Omega \) denote the set of all finitely additive dispersion free states on \( B \). For \( a \in B \), let \( \hat{a} := \{ \omega \in \Omega : \omega(a) = 1 \} \). By Stone's theorem, the family \( \hat{a}, a \in B \) forms a field of clopen sets on the (compact topological space) \( \Omega \), which is isomorphic with \( B \) as a Boolean algebra. Write \( \Sigma_0 = \{ \hat{a} : a \in B \} \), and define a function \( \mu : \Sigma_0 \rightarrow [0, 1] \) by \( \mu(\hat{a}) := s(a) \), \( a \in B \). It is easy to see that \( \mu \) is a finitely additive probability measure on \( \Sigma_0 \). Let \( \Sigma \) denote the \( \sigma \)-field of subsets of \( \Omega \) generated by \( \Sigma_0 \). We will show that \( \mu \) extends to a probability measure on \( \Sigma \). According to the well-known theorems in measure theory (cf. [Halmos, 1962, Theorem A, Sec. 13]), we need to show that \( \mu \) is \( \sigma \)-additive on \( \Sigma_0 \), that is, \( \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) \) whenever a sequence \( \{ A_i \}_{i=1}^{\infty} \) of mutually disjoint sets from \( \Sigma_0 \) is such that \( \bigcup_{i=1}^{\infty} A_i \) belongs to \( \Sigma_0 \). So assume that \( \bigcup_{i}^{\infty} \hat{a}_i = \hat{b} \), for a countable set of pairwise disjoint sets \( \hat{a}_i, i \in \mathbb{N} \), and for some \( b \in B \). Then we see that \( \bigvee_{i}^{\infty} a_i \leq b \). If \( \bigvee_{i}^{\infty} a_i < b \), then there is \( \omega \) such that \( \omega(b) = 1 \), and \( \omega(a_i) = 0 \) for all \( i \), which contradicts our suppositions. Therefore \( b = \bigvee_{i}^{\infty} a_i \), and \( \mu(\hat{b}) = s(\bigvee_{i}^{\infty} a_i) = \sum_{i} s(a_i) = \sum_{i} \mu(\hat{a}_i) \). So \( \mu \) extends to a \( \sigma \)-additive measure on the \( \sigma \)-algebra \( \Sigma \) generated by the algebra \( \Sigma_0 \). Obviously, for every \( a \in B \), the mapping \( \omega \mapsto \omega(a) \) is measurable with respect to \( \Sigma \). As a consequence of the previous considerations,

\[
\int_{\Omega} \omega(a) d\mu(\omega) = \int_{\hat{a}} d\mu(\omega) = \mu(\hat{a}) = s(a).
\]

**THEOREM 135.** [Deliyannis, 1971] Let \((L, S)\) be a system. Then \((L, S)\) admits quasi hidden variables if and only if it is a subsystem of the system associated with a Boolean \( \sigma \)-algebra.

**Proof.** Assume that \( \Omega \) is a generating system of finitely additive states, and let \( \mathcal{N} \) be the \( \sigma \)-ideal of subsets of \( \Omega \) on which all measures associated with the states
$s \in S$ vanish. We define the mapping $a \mapsto \hat{a}$, where $\hat{a}(\omega) = \omega(a)$, and observe that orthocomplements are preserved. Moreover, $a \leq b \iff \hat{a} \leq \hat{b}$ (the latter ordering is pointwise). Let $a = \bigvee a_i$, where $(a_i)_i$ is a sequence of pairwise orthogonal elements. So we have $s(a) = \sum s(a_i)$. Since each $\omega \in \Omega$ is finitely additive, we have

$$\sum \hat{a}_i(\omega) \leq \hat{a}(\omega).$$

But

$$\int_{\Omega} \hat{a}(\omega) - \sum \hat{a}_i(\omega) d\mu_s(\omega) = 0$$

for all $\mu_s, s \in S$ which, together with the above inequality yields $\hat{a} = \sum \hat{a}_i$ modulo $\mathcal{N}$.

Further, let each $\omega$ be dispersion free so that $\hat{a}$ is a characteristic function of a subset of $\Omega$. Let $a^*$ denote the class $\hat{a}$ modulo $\mathcal{N}$. Then $*$ is an isomorphism which sends $L$ into the Boolean algebra $\mathcal{B}$ of subsets of $\Omega$ modulo $\mathcal{N}$. Since all measures which correspond to states in $S$ vanish on $\mathcal{N}$, they produce measures on $\mathcal{B}$, so that $s(a) = \mu_s(a^*)$.

Conversely, if $(L, S)$ is a subsystem of a system associated with a Boolean $\sigma$-algebra, then the result follows by Lemma 134.

In [Deliyannis, 1971, Theorem 5] it is proved that a necessary and sufficient condition for the isomorphism $a \mapsto a^*$ to preserve an existing infimum (supremum) in $L$ is the Jauch-Piron condition for this particular pair to be satisfied in all states generated by $\Omega$. Indeed, all that we need to prove is that letting $c = a \wedge b$ so that $\hat{c} \subseteq \hat{a} \wedge \hat{b} = \{\omega : \hat{a}(\omega)\hat{b}(\omega) = 1\}$, we have $\hat{a} \wedge \hat{b} - \hat{c} \in \mathcal{N}$.

But for every probability measure $\mu$ on $\Sigma$, the map

$$m : x \mapsto \frac{1}{\mu(\hat{a} \wedge \hat{b})} \int_{\hat{a} \wedge \hat{b}} \hat{c}(\omega) d\mu(\omega)$$

is a state on $L$. Since $\hat{a}, \hat{b}$ are 1 on $\hat{a} \wedge \hat{b}$, we have $m(a) = m(b) = 1$, so $m(c) = 1$ as well. But $\hat{c}$ vanishes outside $\hat{a} \wedge \hat{b}$, so that

$$\mu_s(c) = \int_{\Omega} \hat{c}(\omega) d\mu_s(\omega)$$

$$= \int_{\hat{a} \wedge \hat{b}} \hat{c}(\omega) d\mu_s(\omega)$$

$$= \mu_s(\hat{a} \wedge \hat{b}).$$

Therefore $\mu_s(\hat{a} \wedge \hat{b} - \hat{c}) = 0$ for all $\mu_s, s \in S$. In particular, if all finitely additive states in $\Omega$ satisfy the Jauch-Piron property for $a, b$, the result follows.

Notice that both Examples (1a), (1b) admit quasi hidden variables. On the other hand, in Example (2) for $H$ separable of dimension at least 3, there are no finitely additive dispersion free states on $L$ ([Alda, 1981]). Therefore $(L, S)$ does not admit quasi hidden variables.
12 BELLMINEQUALITIES

In this section we assume that \( L \) is a lattice logic, and \( s \) is a finitely additive state on \( L \). We will consider the following inequalities:

1. \( s(a) + s(b) - s(a \land b) \leq 1 \)
2. \( s(b) + s(c) \geq s(a \land b) + s(b \land c) + s(c \land d) - s(a \land d) \)
3. \( s(a) + s(b) + s(c) - s(a \land b) - s(a \land c) - s(b \land c) \leq 1 \)
4. \( (a \land b) + s(b \land c) + s(c \land d) - s(a \land d) - s(b) - s(c) \geq -1 \)

The inequalities (1) and (3) are called the inequalities of Bell-Wigner type, the inequalities (2) and (4) are called the inequalities of Clauser-Horne type [Pitowskii, 1989]. All these inequalities are satisfied by any finitely additive state on a Boolean algebra.

Let \( s \) be a finitely additive state on a logic (in fact, all definitions and results apply well for an OML, thus we do not explicitly need the \( \sigma \)-complete property of \( V \) in \( L \)). Recall that the \( a \) \( s \) is

(i) subadditive if
\[
s(a \lor b) \leq s(a) + s(b) \quad \forall a, b \in L,
\]

(ii) a valuation if
\[
s(a \lor b) + s(a \land b) = s(a) + s(b) \quad \forall a, b \in L,
\]

(iii) Jauch-Piron if \( s(a) = 1 \) and \( s(b) = 1 \) implies \( s(a \land b) = 1 \) (equivalently, \( s(a) = 0 \) and \( s(b) = 0 \) implies \( s(a \lor b) = 0 \)).

Observe that the following implications hold: (ii) \( \Rightarrow \) (iii) and (ii) \( \Leftrightarrow \) (i). If \( s \) is dispersion free, then (i), (ii) and (iii) are equivalent.

Recall that \( a, b \in L \) are compatible if \( a = (a \land b) \lor (a \land b') \), written \( a \leftrightarrow b \). Recall also that \( L \) is a Boolean algebra iff \( a \leftrightarrow b \) holds for all \( a, b \in L \). The element \( \text{com}(a, b) = (a \land b) \lor (a' \land b) \lor (a \land b') \lor (a' \land b') \) is called the lower commutator of \( a, b \), and \( a \leftrightarrow b \) iff \( \text{com}(a, b) = 1 \). The upper commutator is defined by \( \text{com}(a, b) = (a \lor b) \land (a \lor b) \land (a \lor b') \land (a \lor b') = \text{com}(a, b) \).

Recall finally that by the Foulis-Holland theorem, if \( a, b, c \) are elements of \( L \) such that \( a \leftrightarrow b, a \leftrightarrow c \), then \( (a, b, c) \) is a distributive triple, that is, \( x_1 \land (x_2 \lor x_3) = (x_1 \land x_2) \lor (x_1 \land x_3), x_i \in \{a, b, c\}, i \leq 3 \).

**Proposition 136.** [Pulmannová and Majerník, 1992] Let \( L \) be a lattice logic and \( s \) be a finitely additive state on \( L \). The following statements are equivalent: (i) \( s \) is subadditive; (ii) \( s \) is a valuation; (iii) the inequalities (1) hold.
Proof. (i) $\Rightarrow$ (ii): Define $a\Delta b := (a \lor b) \land (a \land b)'$. Using the Foulis-Holland theorem and the fact that $(a' \lor b') \leftrightarrow a$ and $(a' \lor b') \leftrightarrow b$, we obtain

$$(a \lor b) \land (a' \lor b') = (a \land (a' \lor b')) \lor (b \land (a' \lor b')).$$

By the property (i),

$$s(a\Delta b) = s((a \land (a' \lor b')) \lor (b \land (a' \lor b')))$$

$$\leq s(a \land (a' \lor b')) \lor (b \land (a' \lor b'))$$

$$= s(a) - s(a \land b) + s(b) - s(a \land b).$$

This entails that $s(a\Delta b) = s(a \lor b) - s(a \land b) \leq s(a) + s(b) - s(a \land b)$, i.e., $s(a \lor b) + s(a \land b) \leq s(a) + s(b)$. Replacing $a, b$ by $a', b'$, we obtain the converse inequality, and hence $s$ is a valuation.

(ii) $\Rightarrow$ (iii) If $s$ is a valuation, then $s(a) + s(b) - s(a \land b) = s(a \lor b) \leq 1$, hence the inequalities (1) hold.

(iii) $\Rightarrow$ (i): If the inequality (1) holds, then for very $a, b$, $s(a') + s(b') - s(a' \land b') \leq 1$, which yields $1 - s(a) + 1 - s(b) - (1 - s(a \lor b)) \leq 1$, hence $s(a \lor b) \leq s(a) + s(b)$.

We now study the inequalities of type (2). Following [Pykacz and Santos, 1991], we consider the function

$$S(a, b) = s(a) + s(b) - 2s(a \land b).$$

It can be checked by a direct computation that the inequalities (2) are equivalent to the inequalities

$$S(a, b) + S(b, c) + S(c, e) \geq S(a, e),$$

These inequalities hold iff $S$ is a metric on $L$. Put $d(a, b) := s(a\Delta b)$. It is easy to check that $s$ is a valuation iff $S = d$.

PROPOSITION 137. Let $L$ be a lattice logic and let $s$ be a (finitely additive) state on $L$. Then $s$ is a valuation iff $\rho = s(a\Delta b)$ is a metric on $L$.

Proof. First we prove that for any $a, b, c \in L$, $a\Delta b \leq (a\Delta c) \lor (c\Delta b)$. Put $e = a \land b \land c$. Clearly $a \land (a \land b)' \leq (a \lor c) \land e'$. Using orthomodularity, we prove that $(a \lor c) \land e' \leq (a\Delta c) \lor (c\Delta b)$ (see [Pulmannová and Majerník, 1992] for the details). Similarly, $b \land (a \land b)' \leq (b \lor c) \land e' \leq (a\Delta c) \lor (c\Delta b)$. Hence

$$a\Delta b = (a \land (a \land b)') \lor (b \land (a \land b)')$$

$$\leq (a\Delta c) \lor (c\Delta b).$$

Now $s$ is a valuation iff it is subadditive, which yields $\rho(a, b) \leq \rho(a, c)\rho(b, c)$, hence $\rho$ is a metric.

Conversely, if $\rho$ is a metric, then the inequality $\rho(a, b) \leq \rho(a, 0) + d(0, b)$ yields $s(a \lor b) - s(a \land b) \leq s(a) + s(b)$. Replacing $a, b$ by $a', b'$, respectively, we see that the converse inequality holds, and hence $s$ is a valuation.
The above considerations yield the following result.

**THEOREM 138.** Let $L$ be a lattice logic and let $s$ be a (finitely additive) state on $L$. The following statements are equivalent: (i) the inequalities (2) hold; (ii) $S$ is a metric; (iii) $S = \rho$; (iv) $s$ is a valuation; (v) $\rho$ is a metric; (vi) the inequalities (1) hold.

**Proof.** We have already seen that the inequalities (2) hold iff $S$ is a metric. If $S$ is a metric, then $S(a,b) \leq S(a,a\lor b) + S(a\lor b,b)$ implies $s(a) + s(b) - 2s(a\land b) \leq s(a) + s(a\lor b) - 2s(a) + s(a\lor b) + s(b) - 2s(b)$. Consequently, $s(a) + s(b) - s(a\land b) \leq s(a\lor b)$. Replacing $a, b$ by $a', b'$, this yields the converse inequality. Hence $S = \rho$, and $\rho$ is a metric. The remaining statements follow by Proposition 136 and Proposition 137.

**THEOREM 139.** Let $L$ be a lattice logic and let $S$ be an order-determining set of states on $L$. If every state $s \in S$ satisfies Bell inequalities of the type (1), then $L$ is a Boolean algebra.

**Proof.** By Theorem 138, every state $s \in S$ is subadditive. It suffices to prove that $a \leftrightarrow b$ for all $a, b \in L$. By subadditivity we obtain $s((a \lor b) \land b') = s(a \lor b) - s(b') \leq s(a)$ for every $s \in S$, and since $S$ is order-determining, this gives us $(a \lor b) \land b' = a \land b'$. By orthomodularity, $b' = (a' \land b') \lor (a \lor b) \land b' = (a' \land b') \lor (a \land b')$, which entails that $b' \leftrightarrow a$. As this holds for all $a, b \in L$, we conclude that $L$ is a Boolean algebra.

**COROLLARY 140.** If the system $(L,S)$, where $L$ is a lattice logic, satisfies the inequalities (1) for every $s \in S$, then $(L,S)$ admits quasi hidden variables.

The converse need not be true. Let $(B_1, S_1)$ and $(B_2, S_2)$ be the systems associated with Boolean $\sigma$-algebras, and let $L := B_1 + B_2$ denote their horizontal sum. Then $B_1 + B_2$ is a lattice logic, where the supremum $\vee a_i$ corresponds to the supremum taken in $B_j$ if all $a_i$'s belong to $B_j$, $j = 1, 2$, otherwise the supremum is 1. If $s$ is a state on $L$, then the restrictions $s_j := s/B_j$, $j = 1, 2$ are states on $B_j$, $j = 1, 2$, respectively, and $s$ corresponds to the couple $(s_1, s_2)$, where $s(a) = s_j(a)$ iff $a \in B_j$. Conversely, if $(p_1, p_2)$ is a couple of states on $B_j$, $j = 1, 2$, we can form a state $p$ on $L$ by defining $p(a) = p_j(a)$ iff $a \in B_j$, $j = 1, 2$. It can be easily seen that a state $s = (s_1, s_2)$ on $L$ is subadditive, hence Jauch-Piron, iff both $s_1$ and $s_2$ are subadditive, and at least one of them is faithful. Indeed, if neither of $s_1, s_2$ is faithful, then there are elements $a_1 \in B_1, a_2 \in B_2$, both different from 1 with $s_1(a_1) = 1, s_2(a_2) = 1$. But $a_1 \land a_2 = 0$, which contradicts Jauch-Piron property.

It follows that $L$ has no finitely additive dispersion free Jauch Piron state unless one of $B_1, B_2$ is $\{0, 1\}$. But $L$ admits quasi hidden variables. Indeed, consider the tensor product $B_1 \otimes B_2$. The set of product states $s_1 \otimes s_2(a_1 \otimes a_2) = s_1(a_1)s_2(a_2)$ is order-determining for $B_1 \otimes B_2$ (see, e.g., [Aerts and Daubechies, 1978]). Moreover, there is an imbedding $j : B_1 + B_2 \to B_1 \otimes B_2$ such that $j(a) = a \otimes 1$ if $a \in B_1$.

\footnote{Recall that a state $s$ is faithful on $L$ if $s(a) = 1$ iff $a = 1$.}
and \( j(a) = 1 \otimes a \) if \( a \in B_2 \). Further, the states \((s_1, s_2), s_i \in S_i, i = 1, 2\) are in a one-to-one correspondence with the product states \( s_1 \otimes s_2 \). It follows that \( j \) is a system imbedding. An interesting example of this type was found in [Zisis, 2000], where the observables of position \( Q \) and momentum \( P \) in the standard quantum mechanics are considered. Since \( Q \) and \( P \) are complementary ([Pták and Pulmannová, 1991]), it can be shown that the smallest sublogic of \( L(H) \) containing their ranges \( B_1 := \mathcal{R}(Q) \) and \( B_2 := \mathcal{R}(P) \) is isomorphic to the horizontal sum of \( B_1 \) and \( B_2 \).

Now let us consider the inequalities of type (3), i.e., for a state \( s \) and all \( a, b \in L \), let us suppose that

\[
\begin{align*}
\text{If we put } c = 0, \text{ we obtain the inequality (1), hence (3) implies that } s \text{ is subadditive.}
\end{align*}
\]

Recall that a subset \( J \) of a lattice logic \( L \) is a p-ideal if \( (i) \) \( a, b \in J \) imply \( a \lor b \in J \); \( (ii) \) \( a \in J, b \in L \) imply \( (a \lor b') \land b \in J \). There is a one-to-one correspondence between the set of all p-ideals and the set of all congruence relations on \( L \). If \( \theta \) is the congruence corresponding to \( J \), then \( a \equiv b(\theta) \) iff \( a\Delta b \in J \), and \( J = \{a \in L : a \equiv 0(\theta)\} \). Let \( \bar{a} \) denote the congruence class corresponding to \( a \in L \), and let \( L/J \) denote the set of all congruence classes. Then \( L/J \) (the quotient of \( L \) corresponding to \( J \)) is an OML, and the mapping \( \phi : L \to L/J \), \( \phi(a) = \bar{a} \) (the quotient mapping) is a \( \sigma \)-homomorphism.

Let \( J_c \) be the smallest p-ideal which contains the set \( \{\text{com}(a, b) : a, b \in L\} \), where \( \text{com}(a, b) = (a \lor b) \land (a' \lor b') \land (a \lor b') \land (a' \lor b') \) is the upper commutator of \( a, b \). It is well-known that the quotient \( L/J_c \) is a Boolean algebra. Moreover, by [Marsden, 1970],

\[
(10) \quad J_c = \{c \in L : c \leq \bigvee_{i \leq n} (\text{com}(a_i, b_i)), a_i, b_i \in L, i \leq n\}.
\]

Another characterization of Boolean quotients can be obtained by means of prime ideals. Recall that a proper lattice ideal \( J \) is prime if \( a \land b \in J \) implies \( a \in J \) or \( b \in J \). In a lattice logic, an ideal \( J \) is prime iff for all \( a \in L \), either \( a \in J \) or \( a' \in J \). It is straightforward to see that a proper ideal \( J \) of \( L \) is prime iff \( J \) is a p-ideal and the quotient \( L/J \) is \( \{0, 1\} \). Let \( J \) be a prime ideal of an OML \( L \). Define \( p : L \to \{0, 1\} \) by \( p(a) = 0 \) if \( a \in J \) and \( p(a) = 0 \) otherwise. If \( a \perp b \), and \( a \notin J \), then we must have \( b \in J \). This yields \( p(a \lor b) = p(a) + p(b) \). Moreover, \( p(a) = 0 = p(b) \) implies \( p(a \lor b) = 0 \), hence \( p \) is a dispersion free finitely additive Jauch-Piron state (valuation) on \( L \). Conversely, if \( s \) is any dispersion free valuation on \( L \), then \( s^{-1}(0) \) is a prime ideal of \( L \). Recall that for any ideal \( J \), the quotient \( L/J \) is Boolean iff \( J \) is the intersection of all prime ideals that contain it [Kalmbach, 1983]. Consequently, \( J_c \) is the intersection of all prime ideals of \( L \).

Evidently, the notions of a p-ideal and prime ideal make sense also for \( \sigma \)-ideals, i.e. the ideals \( J \) such that \( a_i \in J \) for all \( i \in \mathbb{N} \) implies \( \bigvee_{i \in \mathbb{N}} a_i \in J \). The quotient of a lattice logic with respect to a \( \sigma \) p-ideal is a lattice logic. Obviously, the quotient \( L/J \) is a Boolean \( \sigma \)-algebra if the ideal \( J \) contains all upper commutators.
Recall that a state \( s \) on \( L \) is a superposition [Varadarajan, 1985] of a set \( S \) of states on \( L \) if for any \( a \in L \), \( t(a) = 0 \) \( \forall t \in S \) implies \( s(a) = 0 \). We then have the following characterization of the inequalities of type (3).

**Theorem 141.** Let \( (L, S) \) be a system, where \( L \) is a lattice logic. Denote by \( J_c \) the ideal generated by all upper commutators in \( L \). Let \( s \in S \). The following statements are equivalent:

(i) the inequalities (3) hold;

(ii) \( s(\overline{a \land b}) = 0 \) for all \( a, b \in L \);

(iii) \( s/J_c = 0 \), where \( s/J_c \) denotes the restriction of the state \( s \) to the ideal \( J_c \);

(iv) \( s \) is a superposition of a set of finitely additive dispersion free Jauch-Piron states on \( L \);

(v) there is a Boolean \( \sigma \)-algebra \( B \), a surjective \( \sigma \)-homomorphism \( \phi : L \to B \), and a state \( \tilde{s} \) on \( B \) such that \( \tilde{s} \circ \phi = s \);

(vi) the inequalities (4) hold.

**Proof.** (i) \( \implies \) (ii): Assume that the inequalities (3) hold, and put \( c = a' \). By an easy computation we obtain \( s(b) = s(a \land b) + s(a' \land b) \). Consequently,
\[
1 = s(b) + s(b') = s((a \land b) \lor (a \land b')) + (a' \land b') = s(s(a) + s(s(a' \land b'))).
\]
The latter equality yields \( s(\overline{a \land b}) = 0 \).

(ii) \( \implies \) (iii): Assume that (ii) hold. Then for all \( a, b \) we have \( s(a \lor b) = s((a \lor b) \land a) + s((a \lor b) \land a') = s(a) + s((a \lor b) \land a') \lor (a' \land b') = s(a) + s((a \lor b) \land a') \lor (a' \land b') \lor s(a') = s(a) + s(s(a') + s(a') \land b') \leq s(a) + s(b) \). This proves that \( s \) is subadditive. Using (10), we obtain for any \( c \in J_c \), \( s(c) \leq s(\overline{\bigcup_{i \leq n} \overline{a_i}}) \leq \bigcup_{i \leq n} s(\overline{a_i}) = 0 \).

(iii) \( \implies \) (iv): We have \( s^{-1}(0) \supseteq J_c \), and \( J_c \) is the intersection of all prime ideals of \( L \). Consequently, \( s^{-1}(0) \supseteq \bigcup s^{-1}(0) \), where the intersections goes over all finitely additive dispersion free Jauch-Piron states on \( L \). Hence \( s \) is a superposition of all \( s_{a_i} \).

(iv) \( \implies \) (v): If \( s \) is a superposition of a set \( I \) of finitely additive dispersion free Jauch-Piron states, then \( s^{-1}(0) \supseteq \bigcup_{a \in I} s^{-1}(0) \), and every \( s^{-1}(0) \) is a prime ideal on \( L \). Hence \( J = \bigcup_{a \in I} s^{-1}(0) \) is a p-ideal, and \( J \supseteq J_c \). Let \( \tilde{J} \) be the smallest \( \sigma \)-p ideal containing \( J \). Then \( \tilde{J} \) contains all commutators, and therefore \( B := L/\tilde{I} \) is a Boolean \( \sigma \)-algebra. If \( \phi : L \to B \) is the quotient mapping, then \( \phi \) is a surjective \( \sigma \)-homomorphism. In addition, the mapping \( \tilde{S} : B \to [0, 1] \) defined by \( \tilde{s}(\phi(a)) = s(a) \) is a \( \sigma \) additive state on \( B \).

(v) \( \implies \) (vi): Since \( \tilde{s} \circ \phi = s \) and the inequalities (4) hold for \( \tilde{s} \) on \( B \), it easily follows that the inequalities (4) hold for \( s \) on \( L \).

(vi) \( \implies \) (i): Putting \( d = a \) in the inequalities (4), we obtain inequalities (3).
The following theorem shows that there are close connections between Bell inequalities of type (3) and quasi-hidden variables.

**THEOREM 142.** Let $L$ be a lattice logic, and $s$ be a ($\sigma$-additive) state on $L$. The state $s$ is generated by finitely additive dispersion free Jauch-Piron states on $L$ if and only if one of the conditions of Theorem 141 is satisfied.

**Proof.** If $s$ is generated by the set $\Omega$ of all finitely additive dispersion free Jauch-Piron states on $L$, then there is a $\sigma$-algebra $\Sigma$ of subsets of $\Omega$, which makes the mappings $\omega \mapsto \omega(a)$, $a \in L$, measurable, and there is a probability measure $\mu$ on $\Sigma$ such that for every $a \in L$,

$$s(a) = \int \omega(a) \, d\mu(\omega).$$

It is clear that the equality $\omega(a) = 0$ for all $\omega \in \Omega$ implies $s(a) = 0$. Hence $s$ is a superposition of $\Omega$.

Conversely, suppose condition (v) of Theorem 141 holds. Let $\bar{\Omega}$ be the set of all dispersion free finitely additive states on $B$, and let $\bar{s}$ be the state on $B$ corresponding to $s$. Since $\bar{s}$ is a probability measure on $B$, by Lemma 134, or by applying the Choquet-Bishop-de Leeuw theorem [Phelps, 1966],

$$\bar{s}(\phi(a)) = \int_{\bar{\Omega}} \bar{\omega}(\phi(a)) \, d\mu(\bar{\omega})$$

for a suitable probability measure $\bar{\mu}$.

This yields

$$s(a) = \int_{\omega} \Omega(\omega) \, d\mu(\omega),$$

where $\Omega$ is the set of the finitely additive dispersion free states $\omega$ on $L$ defined by $\omega(a) = \bar{\omega}(\phi(a))$, $a \in L$. Thus $\mu$ is the measure on $\Omega$ induced by $\bar{\mu}$.

**Remarks:**

1. The Bell-type inequalities presented in the previous section were generalized in [Dvurečenskij and Länger, 1995] in the following way: By a *Bell-type inequality of order $n$* is understood any inequality of the type

$$0 \leq \sum_{I \subseteq N} f(I) s(\bigwedge_{i \in I} a_i) \leq 1$$

which holds for a (finitely additive) state $s$ on a logic $L$ (and for all $a_1 \ldots, a_n \in L$), where $f(I)$ is a real coefficient, $s(\bigwedge_{i \in I} a_i)$ is a correlation (or a joint probability) of the set of events $\{a_i : i \in I\}$ in the state $s$, $a_1, \ldots, a_n \in L$, and $N := \{1, 2, \ldots, n\}$. In other words, the function $S_f(s) := \sum_{I \subseteq N} f(I) s(\bigwedge_{i \in I} a_i)$ is a linear combination of $s(\bigwedge_{i \in I} a_i)$, and $s(\bigwedge_{i \in I} a_i)$ can be represented as a joint probability of $a_i (i \in I)$ in the state $s$. In [Beltrametti and Mączyński, 1991] and [Länger and Mączyński, 1995], it has been proved that the inequalities (11) are equivalent to the inequalities...
(12) \[ 0 \leq \sum_{I \subseteq K} f(I) \leq 1 \forall K \subseteq N \]

for every state \( s \) on a Boolean algebra. It was shown in [Dvurečenskij and Länger, 1995] that Bell-type inequalities of order 2 are equivalent with the inequalities (1) (resp. (2)), and the inequalities of order 3 are equivalent with the inequalities (3) (resp. (4)), while higher order Bell-type inequalities are equivalent with Bell-type inequalities of the type (3).

2. Let us finally step into the realm of universal algebra. Let us see how notions of quantum logical theory may be linked with universal algebra. Let \( OML \) denote the variety of orthomodular lattices. Recall that by a famous result of universal algebra [Grätzer, 1986], a subclass \( Q \) of the variety \( OML \) is a quasivariety if and only if \( Q \) is closed under the formation of subalgebras, products, ultraproducts, isomorphic algebras and it contains the trivial algebra. A quasivariety \( Q \) is a variety if it is closed under the formation of epimorphic images. It should be noted that several varieties and quasivarieties of OMLs related to quantum logical theory have been considered in [Binder and Navara, 1987; Godowski, 1981; Carrega et al., 2000; De Simone et al., 2001; Bruns et al., 1990; Matoušek, 2004; Mayet, 1986; Mayet et al., 2000; Haviar and Konôpka, 2000].

For \( x_1, x_2, \ldots, x_n, y_1, y_1, \ldots, y_n \), let \( t_c \) be a term of \( 2n \) variables defined as follows:

\[ t_c(x_1, x_2, \ldots, x_n, y_1, y_1, \ldots, y_n) := \bigvee_{i=1}^{n} \overline{\text{com}}(x_i, y_i). \]

Put

\[ NBQ := \{ L \in OML : \forall x_1, x_2, \ldots, x_n, y_1, y_1, \ldots, y_n \in L : t_c L(x_1, x_2, \ldots, x_n, y_1, y_1, \ldots, y_n) \neq 1_L \} \cup I, \]

where \( I \) is the class of all one-point orthomodular lattices, and the indices indicate to which \( L \) the element belongs.

Recall that for an orthomodular lattice \( L \), the following statements are equivalent:

(i) \( L \) has a nonzero Boolean quotient,

(ii) \( L \) has a proper prime ideal,

(iii) there is a subadditive state on \( L \).

Since every prime ideal contains the ideal \( J_c \) defined by (10), it can be easily seen that \( L \in NBQ \) iff one of the equivalent conditions (i)-(iii) is satisfied.

PROPOSITION 143. [D'Andrea and Pulmannová, 1994] (i) every free orthomodular lattice belongs to \( NBQ \). (ii) The variety \( BA \) of Boolean algebras is a proper subclass of \( NBQ \). (iii) The class \( NBQ \) is a proper quasivariety of \( OML \).
Proof. (i) Let $F$ be a free orthomodular lattice over a nonempty set $X$. Let $J_c$ be the commutator p-ideal in $F$. Then the quotient mapping $h : F \rightarrow F/J_c$ maps $F$ into $BA$, and $F/J_c$ is a free Boolean algebra over $h(X)$ ($h(X) \neq \emptyset$).

(ii) Obviously, $BA \subseteq NBQ$. On the other hand, let $F_2$ denote the free orthomodular lattice over the set $\{x, y\}$ [Beran, 1985]. Then $F_2$ belongs to $NBQ$, but it does not belong to $BA$.

(iii) From the definition of $NBQ$, we immediately see that $NBQ$ is characterized by the set of quasiidentities $\forall n \in \mathbb{N}, t_c(x_1, x_2, \ldots, x_n, y_1, y_1, \ldots, y_n) \leq 1 \implies 0 \leq 1$. Therefore $NBQ$ is a quasivariety. To show that $NBQ$ is not a variety, consider the OML $MO_2$ (the horizontal sum of two four-point Boolean algebras), which is a homomorphic image of $F_2$, but it is simple.

Let us denote by the symbol $Bell$ the class of orthomodular lattices $L$ satisfying the following property: $L \in I$ or there is a state $s$ on $L$ such that $(L, s)$ satisfies one of the equivalent conditions of Theorem 141. The proof of the following statement is then easy.

THEOREM 144. The class Bell coincides with the class $NBQ$, hence it is a proper quasivariety in OML.

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