Cyclotomy: From Euler through Vandermonde to Gauss

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The word “cyclotomy” is of Greek origin and means “division of the circle.” As a mathematical term it denotes the subdivision of a full circle line into a given number of equal parts. Consider the unit circle $x^2 + y^2 = 1$ in the Euclidean plane with Cartesian coordinates $(x, y)$. If this circle is divided into $n$ equal parts beginning with the point $(1, 0)$ then the other division points will have coordinates $(\cos 2\pi \cdot k/n, \sin 2\pi \cdot k/n)$ where $k$ runs from 1 to $(n - 1)$. All those points form the edges of a regular $n$-sided polygon.

It is well-known that by means of the imaginary quantity $i := \sqrt{-1}$ one can prove the formula

$$(\cos \alpha + i \cdot \sin \alpha)^n = \cos n\alpha + i \cdot \sin n\alpha$$  \hspace{1cm} (1)

which is usually called de Moivre’s formula. But in the form (1) it is due to Leonhard Euler (1707-1783), see [Euler 1748], cap. VIII. In particular, the $n$ arguments $\alpha = \frac{2\pi \cdot k}{n}$ with $0 \leq k \leq n - 1$ provide us with the $n$ powers $1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}$ of the complex number $\zeta_n := \cos \frac{2\pi}{n} + i \cdot \sin \frac{2\pi}{n}$:

$$
\zeta_n^k = \cos \frac{2\pi \cdot k}{n} + i \cdot \sin \frac{2\pi \cdot k}{n} \quad (0 \leq k \leq n - 1)
$$  \hspace{1cm} (2)

satisfying the equation

$$x^n - 1 = 0.$$  \hspace{1cm} (3)
This means that Eqn. (3) has exactly \( n \) roots which are given in the transcendental form (2) and which are the powers of one of them, namely \( \zeta_n \). For these powers we shall adopt the name \( n \)th roots of unity common today among mathematicians. If the exponent \( i \) is prime to \( n \) then \( \zeta_i \) is called a primitive \( n \)th root of unity since its powers \( 1, \zeta_i, \zeta_i^2, \zeta_i^3, \ldots \) run through all \( n \)th roots of unity.\(^1\)

This way, the geometric problem of cyclotomy was entirely reduced to the algebraic problem of solving the equations \( x^n - 1 = 0 \) in complex numbers. On the other hand, in the 18th century the dominating problem in the theory of equations was still the problem of solving equations “algebraically” or “by radicals” which meant by the extraction of roots. Euler also devoted some papers to this topic and he was very well aware that roots like \( \sqrt[n]{A} \) are only determined up to factors which are \( n \)th roots of unity. These ambiguities led him to investigate the “binomial” equation \( x^n - 1 = 0 \), [Euler 1751], §§ 38-48. It was Euler who succeeded in solving the equations \( x^n - 1 = 0 \) for \( n \leq 10 \) in terms of radicals with indices < \( n \). In his treatise [Vandermonde 1774] Alexandre-Théophile Vandermonde (1735-1796) overcame the difficulties occuring in the case \( n = 11 \) in a truly pioneering way. Eventually, Carl Friedrich Gauss (1777-1855) was the brilliant architect of a fully-fledged cyclotomy theory which among other topics solved the equations (3) by radicals at least for prime number exponents \( n \), [Gauss 1801], Sectio septima.

In the present paper we intend to display the evolution of ideas from Euler to Gauss. Our first section is devoted entirely to Euler, in particular to his solution of \( x^7 - 1 = 0 \) by square and cubic roots. The second section elucidates Vandermonde’s innovations in the theory of equations which parallel the work of Joseph-Louis Lagrange (1736-1813) in many aspects. Vandermonde however far excelled Euler as well as Lagrange by his solution of \( x^{11} - 1 = 0 \). We shall expose a version of “pre-Gaussian” cyclotomy theory based on the ideas and tools of Euler, Vandermonde and Lagrange.\(^2\) Our third section mainly discusses Gauss’s relationship to Vandermonde. On the example of \( x^{17} - 1 = 0 \) we are going to make visible the difference between the cyclotomy theories of these two mathematicians. It is very remarkable that Vandermonde’s ideas also allow us to construct\(^1\)

\(^1\) The primitive \( n \)th roots of unity should not be confused with the primitive roots \( a \equiv 0 \) (mod \( n \)) for prime numbers \( n \) which bear their name since the powers \( 1, a, a^2, a^3, \ldots \) run through all residue classes \( \not\equiv 0 \) (mod \( n \)), in other words, \( a \) mod \( n \) should be a generator of the prime residue class group mod \( n \).

\(^2\) In the conceptual framework of our recent set-theoretic mathematics this version amounts to studying the maximal totally real subfields \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) of \( \mathbb{Q}(\zeta_n) \) first and after that considering the fields \( \mathbb{Q}(\zeta_n) \) as quadratic extensions of \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \).
the regular 17-gon by ruler and compass. Our considerations will be self-contained and entirely based on Vandermonde’s theory of quartic equations and straightforward calculations. This possibility seems to have gone unnoticed until now. This very much begs the question whether Gauss became acquainted with Vandermonde’s work before or after publishing his *Disquisitiones Arithmeticae* and whether Vandermonde could have exerted some effect on Gauss. Our main thesis is that Gauss was not essentially influenced by Vandermonde’s algebraic work in contrast to conjectures formulated by Henri Lebesgue (1875-1941) and adopted or reproduced by other authors without further examination, see [Lebesgue 1940], [Jones 1991], [Waerden], p. 79.

In the end some concluding remarks should show the reader how to link our considerations to Gauss’s theory of cyclotomy.

Comments on mathematical facts in modern terms and notations will, as a rule, not be given in the text but in footnotes.

1. **Euler**

Here we are going to explain in detail how Euler solved the equations $x^n - 1 = 0$ by radicals for $n \leq 10$. We skip the rather easy cases $n \leq 4$ and pass at once to the equation $x^5 - 1 = 0$. Division by $x - 1$ gives the new equation $x^4 + x^3 + x^2 + x + 1 = 0$ which is mirror-symmetric with regard to the middle term $x^2$.

1.1. **Reciprocal Equations**

Equations of this type were already considered by Euler in his paper [Euler 1738], § 10 seq. There he is dealing with mirror-symmetric equations of the general form

$$y^{2n} + a \cdot y^{2n-1} + b \cdot y^{2n-2} + \ldots + p \cdot y^n + \ldots + b \cdot y^2 + a \cdot y + 1 = 0, \quad (4)$$

especially for $n = 2, 3, 4, 5$, and is calling them *reciprocal equations* since they do not change their form when $y$ is replaced by $\frac{1}{y}$.\footnote{It should be emphasized that our construction does not explicitly use a primitive root mod 17 as a generator of all residue classes $\not\equiv 0 \pmod{17}$, contrary to Gauss.}

Now Euler observes that the left-hand side of Eqn. (4) is a product of $n$ quadratic factors

$$y^2 + \alpha \cdot y + 1, y^2 + \beta \cdot y + 1, y^2 + \gamma \cdot y + 1, y^2 + \delta \cdot y + 1, \text{etc.}$$

\footnote{\textit{Aequationes huiusmodi, quae posto $\frac{1}{y}$ loco $y$ formam non mutant, voco reciprocas.} [Euler 1738], § 11.}
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where the coefficients \( \alpha, \beta, \gamma, \delta \), etc. satisfy an equation of degree \( n \) the coefficients of which are nothing but linear combinations of the coefficients of Eqn. (4) with some explicitly known rational integers. The idea behind the factorization of Eqn. (4) can be exposed as follows. Divide Eqn. (4) by \( y^n \) and write the resulting equation in the form

\[
(y^n + y^{-n}) + a \cdot (y^{n-1} + y^{-n+1}) + b \cdot (y^{n-2} + y^{-n+2}) + \ldots + p = 0.
\]

The \( k \)th power sum \( y^k + y^{-k} \) (\( 0 \leq k \leq n \)) is symmetric in \( y \) and \( y^{-1} \), hence a polynomial in the elementary symmetric polynomials \( z := y + y^{-1} \) and \( y \cdot y^{-1} = 1 \) according to the recursive relation

\[
y^{k+1} + \frac{1}{y^{k+1}} = \left( y^k + \frac{1}{y^k} \right) \cdot z - \left( y^{k-1} + \frac{1}{y^{k-1}} \right)
\]

(or by the so-called Girard-Newton formulas for power sums). Hence the auxiliary quantity \( z := y + y^{-1} \) satisfies an equation

\[
z^n + a' \cdot z^{n-1} + b' \cdot z^{n-2} + \ldots + p' = 0 \tag{5}
\]

where the coefficients \( a', b', \ldots, p' \) are linear combinations of \( a, b, c, \ldots, p \) with rational integer coefficients. Denote by \(-\alpha, -\beta, -\gamma, -\delta\), etc. the roots of Eqn. (5), in other words \( z^n + a' \cdot z^{n-1} + b' \cdot z^{n-2} + \ldots + p' = (z + \alpha) \cdot (z + \beta) \cdot (z + \gamma) \cdot (z + \delta) \cdot \ldots \). In view of \( z = y + y^{-1} \) it is obvious that every root of Eqn. (4) satisfies one of the quadratic equations

\[
y^2 + \alpha \cdot y + 1 = 0, \quad y^2 + \beta \cdot y + 1 = 0,
\]

\[
y^2 + \gamma \cdot y + 1 = 0, \quad y^2 + \delta \cdot y + 1 = 0, \quad \text{etc.}
\]

whence one gets the desired factorization of Eqn. (4). Below we shall encounter the reciprocal equations once more in Vandermonde’s work.

1.2. Roots of Unity

After these preparations it is rather easy to solve the reciprocal equation \( x^4 + x^3 + x^2 + x + 1 = 0 \) by radicals. We put \( u := -(x + x^{-1}) \) and obtain \( x^2 + x^{-2} = u^2 - 2 \). In summary \( u^2 - u - 1 = 0 \) with the roots \( p := \frac{1 + \sqrt{5}}{2} \) and \( q := \frac{1 - \sqrt{5}}{2} \). Then one has to solve the two quadratic equations \( x^2 + p \cdot x + 1 = 0 \) and \( x^2 + q \cdot x + 1 = 0 \) but that does not cause any problems, [Euler 1751], § 40. It turns out that all 5th roots of unity are rational functions (with rational coefficients) of \( \sqrt{5} \) and the square root \( \sqrt{-10 + 2 \sqrt{5}} \):

\[
x = \frac{1}{4} \left[ -1 - \delta_1 \sqrt{5} + \delta_2 \sqrt{-10 - \delta_1 2 \sqrt{5}} \right] \tag{6}
\]
with $\delta_1, \delta_2 = \pm 1$, [Euler 1751], § 40.  

Notice that
\[
\sqrt{-10 + 2\sqrt{5}} \cdot \sqrt{-10 - 2\sqrt{5}} = 4\sqrt{5}.
\]

The case $n = 6$ is settled by the factorization
\[
x^6 - 1 = (x^2 - 1) \cdot (x^2 + x + 1) \cdot (x^2 - x + 1),
\]
[Euler 1751], § 41.

The case $n = 8$ can be dealt with by the remark that
\[
x^8 - 1 = (x^2)^4 - 1 = 0,
\]
in other words, we have to extract square roots from the solutions of $x^4 - 1 = 0$. Euler’s formulas show that all 8th roots of unity are rational functions of $\sqrt{-1}$ and $\sqrt{2}$ (with rational coefficients). In a similar way, Euler solves the equation $x^{10} - 1 = 0$ by extracting cubic roots from the third roots of unity. Last but not least the equation $x^{10} - 1 = 0$ is solved via the factorization $x^{10} - 1 = (x^5 - 1) \cdot (x^5 + 1)$, [Euler 1751], § 48.

The remaining case $n = 7$ requires a great amount of calculations. First Euler follows the general approach to reciprocal equations and then simplifies the resulting expressions of the 7th roots of unity carrying out some further subtle calculations. He factorizes the equation
\[
x^7 - 1 = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0
\]
into three quadratic equations
\[
x^2 + p \cdot x + 1 = 0, \quad x^2 + q \cdot x + 1 = 0, \quad x^2 + r \cdot x + 1 = 0
\]
where $p, q, r$ are the roots of the cubic equation
\[
F(u) := u^3 - u^2 - 2u + 1 = 0.
\]

Now he applies Cardano’s formula to this equation and obtains the explicit expressions (in our abbreviations)
\[
p = \frac{1}{3} \left[ 1 + \sqrt[3]{A} + \sqrt[3]{A'} \right], \quad q = \frac{1}{3} \left[ 1 + \rho \cdot \sqrt[3]{A} + \rho^2 \cdot \sqrt[3]{A'} \right],
\]
\[
r = \frac{1}{3} \left[ 1 + \rho^2 \cdot \sqrt[3]{A} + \rho \cdot \sqrt[3]{A'} \right]
\]
with
\[
A = 10, \quad A' = -10 + 2\sqrt{5}, \quad \rho = e^{\frac{2\pi i}{3}}.
\]

\[5\] In today’s terms we have $\mathbb{Q}(\zeta_5) = \mathbb{Q}(\sqrt{5})(\sqrt{-10 + 2\sqrt{5}})$ according to Euler. The cyclotomic extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is cyclic of degree 4 which can be proved by means of Gauss’s theory.
\[ \rho = \frac{-1 + \sqrt{-3}}{2}, \quad A = 7 \cdot \frac{-1 + 3 \cdot \sqrt{-3}}{2}, \]
\[ A' = 7 \cdot \frac{-1 - 3 \cdot \sqrt{-3}}{2}, \quad \sqrt{A} \cdot \sqrt{A'} = 7. \]

After that Euler goes on to solve Eqns. (7). For instance, the first equation in (7) has the roots
\[ x = \frac{-p \pm \sqrt{p^2 - 4}}{2}. \] (10)
Substituting \( q, r \) instead of \( p \) we obtain the roots of the other two equations. At this point there arises the problem of how to extract square roots from expressions which in their turn are sums of certain cubic radicals. Euler tackles the explicit calculation of the square roots
\[ \sqrt{p^2 - 4}, \quad \sqrt{q^2 - 4}, \quad \sqrt{r^2 - 4}, \]
putting \( v := \sqrt{u^2 - 4} \) and indicating the cubic equation for \( v^2 \), i.e. the equation of degree 6 for \( v \):
\[ G(v) := v^6 + 7v^4 + 14v^2 + 7 = 0 \]
with the six roots \( \pm \sqrt{p^2 - 4}, \pm \sqrt{q^2 - 4}, \pm \sqrt{r^2 - 4} \). This equation splits up into two cubic equations according to
\[ (v - \sqrt{p^2 - 4})(v - \sqrt{q^2 - 4})(v - \sqrt{r^2 - 4}) =: v^3 + p'v^2 + q'v + r' = 0 \]
and
\[ (v + \sqrt{p^2 - 4})(v + \sqrt{q^2 - 4})(v + \sqrt{r^2 - 4}) =: v^3 - p'v^2 + q'v - r' = 0. \]
But Euler’s choice of a suitable triple \( (\sqrt{p^2 - 4}, \pm \sqrt{q^2 - 4}, \pm \sqrt{r^2 - 4}) \) out of 4 possibilities ones looks quite arbitrary and calls for a systematic procedure. The crucial point is that this choice can be made such that the coefficients \( p', q', r' \) take the values
\[ p' = \sqrt{-7}, \quad q' = 0, \quad r' = \sqrt{-7} \]
which results from the comparison of coefficients in the product of the last two equations and in (11). Eventually these coefficients require a new quadratic irrationality only. 6 To sum up we have two cubic equations
\[ v^3 \pm \sqrt{-7} \cdot v^2 \pm \sqrt{-7} = 0 \]
---

6 The right factorization of \( G(v) \) is closely tied up to the quadratic residues mod 7 and will not be discussed here in detail. It seems that Euler was not aware of this fact. Anyway we have \(-r'^2 = 7, \sqrt{p^2 - 4} \cdot \sqrt{q^2 - 4} \cdot \sqrt{r^2 - 4} = -\sqrt{-7}. \) With regard to Eqns. (10) one concludes from there that \( \sqrt{-7} \) is a linear combination of 7th roots of unity with rational integer coefficients. This remarkable fact was generalized by Gauss to all prime exponents \( n \) instead of 7, [Gauss 1801], art. 356. In recent terms it means the embedding of all quadratic number fields in cyclotomic fields. See also our subsection 2.5 after Eqn. (58).
and Cardano’s formula gives us six values of \(v\). These values turn out to be rational functions (with rational coefficients) of \(\sqrt{-3}\), \(\sqrt{-7}\) and the cubic radicals \(\sqrt[3]{B}, \sqrt[3]{B'}\) with

\[
B = \alpha^2 \sqrt{-7}, \quad B' = \alpha'^2 \sqrt{-7}, \quad 3\sqrt[3]{B} \cdot 3\sqrt[3]{B'} = -7
\]

and

\[
\alpha = \frac{-1 + 3\sqrt{-3}}{2}, \quad \alpha' = \frac{-1 - 3\sqrt{-3}}{2}, \quad \alpha \cdot \alpha' = 7 = -7. \tag{1}
\]

Furthermore it is obvious how to remove the factor \(\sqrt{-7}\) from \(B\) and \(B'\):

\[
\sqrt[3]{B} = \frac{\alpha}{\sqrt{-7}}, \quad \sqrt[3]{B'} = \frac{\alpha'}{\sqrt{-7}}, \quad \sqrt[3]{\alpha} \cdot \sqrt[3]{\alpha'} = 7.
\]

Euler’s definite formulas show that the 7th roots of unity are rational functions (with rational coefficients) of the two quadratic irrationalities \(\sqrt{-3}, \sqrt{-7}\) and a single cubic radical \(\sqrt[3]{7}\alpha\) only. \(\uparrow\)

What can be objected to in Euler’s exposition? As to the 7th roots of unity his formulas are correct but incomplete at one point. He does not mention the relations

\[
\sqrt[3]{A} \cdot \sqrt[3]{A'} = 7, \quad \sqrt[3]{B} \cdot \sqrt[3]{B'} = -7, \quad \sqrt[3]{\alpha} \cdot \sqrt[3]{\alpha'} = 7
\]

which are indispensable in order to obtain the actual roots of the cubic equations in question. However, we don’t hesitate to suppose that Euler had these relations in mind as well, cf. [Euler 1738], § 3-6, [Euler 1770], part II, section 1, § 12.

More generally, in his 1738 and 1751 papers beyond the cubic and quartic equations Euler does not address the important question in full generality of how to restrict the ambiguities in expressions like \(\sqrt[3]{A} + \sqrt[3]{B} + \sqrt[3]{C} + \sqrt[3]{D} + \ldots\) for the roots of an equation of degree \(n\). This question arises in a natural way since every \(n\)th root \(\sqrt[3]{A}\), etc. takes \(n\) values. Only later on in his 1764 paper Euler discusses equations with solutions of the special form \(\omega + A\sqrt[3]{\nu} + B(\sqrt[3]{\nu})^2 + C(\sqrt[3]{\nu})^3 + \ldots + O(\sqrt[3]{\nu})^{n-1}\) and obtains all

\(\uparrow\) In today’s terminology this means that the cyclotomic number field \(\mathbb{Q}(\zeta_7, \sqrt{-3}) = \mathbb{Q}(\zeta_7, \zeta_6) = \mathbb{Q}(\zeta_{42})\) is identical with the field \(\mathbb{Q}(\sqrt{-7}, \sqrt{-3}, \sqrt[3]{7\alpha})\). The Galois extension \(\mathbb{Q}(\zeta_7)/\mathbb{Q}\) is cyclic of degree 6 which can be deduced from Gauss’s theory. Hence this extension is the composite of \(\mathbb{Q}(\sqrt{-7})/\mathbb{Q}\) and a uniquely determined cyclic extension \(L/\mathbb{Q}\) of degree 3. \(L\) is nothing but the splitting field of \(u^3 - u^2 - 2u - 1\), and the “cycle” of explicit relations \(q = p^3 - 2, r = q^3 - 2, p = r^3 - 2\) shows that \(L = \mathbb{Q}(p, q, r) = \mathbb{Q}(p) = \mathbb{Q}(q) = \mathbb{Q}(r)\). We have \(L(\sqrt{-3}) = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{7\alpha})\) which is a cyclic extension of \(\mathbb{Q}\) of degree 6. By the way, the equality \(7 = \alpha \cdot \alpha'\) yields the unique prime factor decomposition of 7 in the euclidean domain \(\mathbb{Z}\{(−1 + \sqrt{-3})/2\}\).
solutions by multiplying \((\sqrt[n]{v})^i\) by the \(i\)th power of a \(n\)th root of unity \((1 \leq i \leq n - 1)\), [Euler 1764], esp. § 13, [Breuer 1921], [Euler 1928], pp. 65-94 (annotations by the editor), [Maistora 1985].

How could one improve on Euler’s exposition using Eulerian tools only? In hindsight it is tempting to describe all occurring quantities in terms of roots of unity. Let \(\zeta\) be a primitive 7th root of unity. Thus we have

\[
p = -\zeta - \zeta^6, \quad q = -\zeta^2 - \zeta^5, \quad r = -\zeta^4 - \zeta^3
\]

(12)

and

\[
p^2 - 4 = (\zeta - \zeta^6)^2, \quad q^2 - 4 = (\zeta^2 - \zeta^5)^2, \quad r^2 - 4 = (\zeta^4 - \zeta^3)^2.
\]

(13)

Now it is rather easy to write down two suitable factors of \(G(v)\):

\[
(v - (\zeta - \zeta^6))(v - (\zeta^2 - \zeta^5))(v - (\zeta^4 - \zeta^3)) = v^3 + \sqrt{-7}v^2 + \sqrt{-7}
\]

(14)

and

\[
(v + (\zeta - \zeta^6))(v + (\zeta^2 - \zeta^5))(v + (\zeta^4 - \zeta^3)) = v^3 - \sqrt{-7}v^2 - \sqrt{-7}.
\]

(15)

The equality

\[
(\zeta - \zeta^6)(\zeta^2 - \zeta^5)(\zeta^4 - \zeta^3) = -\sqrt{-7}
\]

(16)

allows us to solve Eqn. (14) without a repeated application of Cardano’s formula contrary to Euler. Indeed each of the three products

\[
(\zeta - \zeta^6)(\zeta^2 - \zeta^5), \quad (\zeta^2 - \zeta^5)(\zeta^4 - \zeta^3), \quad (\zeta^4 - \zeta^3)(\zeta - \zeta^6)
\]

(17)

is symmetric in \(\zeta, \zeta^{-1}\), hence a polynomial in \(p\) or \(q\) or \(r\) as we want. For instance, we have

\[
(\zeta - \zeta^6)(\zeta^2 - \zeta^5) = (\zeta - \zeta^6)^2(\zeta + \zeta^6) = -(p^2 - 4)p.
\]

(18)

This implies \(\zeta^4 - \zeta^3 = \frac{\sqrt{-7}}{(p^2 - 4)p}\) immediately and it only remains for us to calculate the reciprocal values of \(p\) and \(p^2 - 4\) which is an easy standard exercise.\(^8\) We will further comment on those equations from a more general point of view in our next section on Vandermonde.

After his splendid results published in 1751 Euler was not able to settle the case of the 11th roots of unity. He said that

“indeed the eleven roots of the equation \(x^{11} - 1 = 0\) cannot be calculated with the help of the accompanying equation of degree 5; since its solution is hidden hitherto we should stop here.”\(^9\)

For us now it will be the right moment to pass to Vandermonde’s work and his solution of \(x^{11} - 1 = 0\).

\(^8\) We know the equations (8) and (11) satisfied by \(p\) and \(\sqrt{p^2 - 4}\), resp., and can write \((p^2 - p - 2)p + 1 = 0\) and \((p^2 - 4)^2 + 7(p^2 - 4) + 14)(p^2 - 4) + 7 = 0.\n
\(^9\) At vero radices undecim aequationis \(x^{11} - 1 = 0\) exhiberi non possunt ope aequationis quinque dimensionum; cuius resolutio cum adhuc lateat, hic subsistere debemus. [Euler 1751], § 48.
2. Vandermonde

Without any doubt Vandermonde was an outstanding mathematician of his time, which is confirmed by Lebesgue’s deserving and authoritative biography, [Lebesgue 1940], [Jones 1991]. He published only four mathematical papers among which there were two significant ones. The geometrico-topological paper *Remarques sur des problèmes de situation* (1771) aroused Gauss’s interest and let him speak of “the geometer Vandermonde held in high esteem by me”, [Olbers 1900], p. 103. As far as we know this is the first documented mentioning of Vandermonde by Gauss.

Here we will mainly be concerned with Vandermonde’s extensive *Mémoire sur la résolution des équations* which was read before the Paris Academy in November 1770 but was not published until 1774. Apparently, the British mathematician Edward Waring (1734-1798) was the first who appreciated Vandermonde’s contributions to the theory of equations and praised his acumen, [Waring 1782], Praefatio, pp. XXIV-XXV. More than two decades later Lagrange commented on Vandermonde’s theory of equations rather extensively, [Lagrange 1808], notes XIII, XIV. He said:

“Thus one may say that Vandermonde is the first who had crossed the limits within which the solution of equations of 2 terms was constricted”,

Augustin-Louis Cauchy (1789-1857) in two of his papers on rational functions and permutations referred to Vandermonde as well, [Cauchy 1815a,b]. No less a mathematician than Leopold Kronecker (1823-1891) praised the memoir in the words:

“With Vandermonde’s memoir on the resolution of equations, presented in 1770 to the Parisian Academy, began a new blossoming of algebra; the profundity of the view which is expressed in such clear words as in this work, arouses nothing less than our astonishment.”

Meanwhile, several detailed overviews of Vandermonde’s algebraic work were published, e. g., [Loewy 1918], [Lebesgue 1940], [Wussing 1969], 2. Kap., [Nový 1973], pp. 36-41, [Edwards 1984], § § 15-16, § § 22-23, [Waa-
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den 1985], pp. 77-79, [Tignol 1988], chap. 11. Here we shall concentrate our attention on his theory of equations and its application to cyclotomic equations.

2.1. Resolvents

In the introductory sentences of his treatise Vandermonde mentioned the papers [Euler 1764] and [Bézout 1764] as the most significant ones of the recent past. His own most notable innovation can be described as follows:

“Starting with the well known solution of quadratic and cubic equations, Vandermonde develops general principles upon which the solution of equations may be based. (...) Vandermonde now asks whether the general equation of degree \( n \) can be solved by a similar expression

\[
\frac{1}{n} [x_1 + \cdots + x_n + \sqrt[n]{r_1 x_1 + \cdots + r_n x_n} + \cdots + \sqrt[n]{r_1^{n-1} x_1 + \cdots + r_n^{n-1} x_n} ]
\]

in which \( r_1, \ldots, r_n \) are the \( n \)th roots of unity.” [Waerden 1985], p. 77, cf. [Vandermonde 1774], § VI.

Of course, \( x_1, \ldots, x_n \) denote the roots of the given equation. Vandermonde and Lagrange introduced (or should one say: invented?) expressions like

\[
\Delta^{(i)} := r_1^i x_1 + \cdots + r_n^i x_n, \quad (1 \leq i \leq n),
\]

(19) independently of each other and almost at the same time, [Lagrange 1770-1771], § 69. Nowadays these expressions usually are called “Lagrange resolvents”. The Eqns. (19) can be regarded as a system of linear equations for \( x_1, \ldots, x_n \) which has the solution

\[
x_k = \frac{1}{n} \cdot \sum_{i=1}^{n} r_k^{-i} \cdot \Delta^{(i)} = \frac{1}{n} \cdot \left[ (x_1 + \cdots + x_n) + \sum_{i=1}^{n-1} r_k^{-i} \cdot \Delta^{(i)} \right],
\]

(20)

for \( 1 \leq k \leq n \). For \( n \leq 7 \) the reader will also find these explicit solutions in Vandermonde in §§ VII-X.

Vandermonde and Lagrange alike observed that in (19) the whole sum is multiplied by a \( n \)th root of unity if on the \( x_1, \ldots, x_n \) a suitable cyclic permutation, (say) \( \sigma \), or one of its powers is performed (this permutation depends on the sequence \( (r_1, \ldots, r_n) \)).

Therefore the \( n \)th power of each \( r_k = \rho^{k-1} \) where \( \rho \) denotes a primitive \( n \)th root of unity (\( 1 \leq k \leq n \)) as Lagrange did. Under this assumption \( \sigma \) can be taken as \( x_1 \mapsto x_2, \ldots, x_n \mapsto x_1 \). For odd prime numbers \( n = 2m+1 \) Vandermonde prefers to choose

\[
r_1 = 1, r_2 = \rho, r_3 = \rho^{-1}, r_4 = \rho^2, r_5 = \rho^{-2}, \ldots, r_{2m} = \rho^m, r_{2m+1} = \rho^{-m}, [Vandermonde 1774], § XI.
\]

\[13\]The simplest way to see that is to number the roots \( x_1, \ldots, x_n \) such that \( r_k = \rho^{k-1} \) where \( \rho \) denotes a primitive \( n \)th root of unity (\( 1 \leq k \leq n \)) as Lagrange did. Under this assumption \( \sigma \) can be taken as \( x_1 \mapsto x_2, \ldots, x_n \mapsto x_1 \). For odd prime numbers \( n = 2m+1 \) Vandermonde prefers to choose \( r_1 = 1, r_2 = \rho, r_3 = \rho^{-1}, r_4 = \rho^2, r_5 = \rho^{-2}, \ldots, r_{2m} = \rho^m, r_{2m+1} = \rho^{-m} \), [Vandermonde 1774], § XI.
of the expressions (19) remains unchanged under $\sigma$. We put

$$V^{(i)} := (\Delta^{(i)})^n = (r_1^i x_1 + \ldots + r_n^i x_n)^n \quad (1 \leq i \leq n). \quad (21)$$

In many examples in § XXXVI Vandermonde is “expanding” the $V^{(i)}$’s in sums of simple $\sigma$-invariant components which he calls “partial types” (types partiels) and each of which can be calculated separately. Lagrange’s exposition is much more extensive than Vandermonde’s and explains the important step from (19) to (21) as follows. For any $i$, consider the values of $\Delta^{(i)}$ under all $n!$ permutations of $x_1, \ldots, x_n$. These $n!$ values can be arranged in groups of $n$ values each consisting of $\Delta, \rho \cdot \Delta, \rho^2 \cdot \Delta, \ldots, \rho^{n-1} \cdot \Delta$ where $\rho$ denotes a primitive $n$th root of unity. Let $\Delta'$ run through all $n!$ values of $\Delta^{(i)}$. Then the polynomial $\prod (x - \Delta')$ splits up into factors $\prod_{i=0}^{n-1} (x - \rho^i \Delta) = x^n - \Delta^n$. In other words $\prod (x - \Delta')$ is actually a polynomial in $x^n$ with the roots $(\Delta')^n$. Therefore, the quantity $V^{(i)}$ satisfies an equation of degree $(n-1)!$ the coefficients of which are rational functions of the coefficients of the given equation and the $n$th roots of unity.

Combining Eqns. (20) and (21) we obtain Vandermonde’s and Lagrange’s approach

$$x = \frac{1}{n} \cdot \left[ (x_1 + \ldots + x_n) + \sqrt[n]{V^{(1)}} + \ldots + \sqrt[n]{V^{(n-1)}} \right] \quad (22)$$

where the $n$th roots are to be suitably chosen. Vandermonde calls the right-hand side of (22) a “function which one could say equals any root depending on the meaning attributed to that function.” [Vandermonde 1774, § IV.]

Though for $n > 4$ neither Vandermonde nor Lagrange addresses the question of how to choose the $n$ suitable $(n-1)$-tuples $(\sqrt[n]{V^{(1)}}, \ldots, \sqrt[n]{V^{(n-1)}})$ out of the $n^{n-1}$ possible ones. But they are well aware of the following partial answer to this question: if, in (22), $(\sqrt[n]{V^{(1)}}, \ldots, \sqrt[n]{V^{(n-1)}})$ gives us a solution then $(\rho \cdot \sqrt[n]{V^{(1)}}, \rho^2 \cdot \sqrt[n]{V^{(2)}}, \ldots, \rho^{n-1} \cdot \sqrt[n]{V^{(n-1)}})$ will do as well where $\rho$ denotes any $n$th root of unity, see Eqn. (20) and [Vandermonde 1774, §§ VII-X.] It was only Gauss in case of cyclotomy and Niels Henrik Abel (1802-1829) under more general assumptions who settled that question based on the remark that the products $\Delta^{(i)} \cdot (\Delta^{(1)})^{n-i}, (2 \leq i \leq n-1)$ as well as the $V^{(i)}$’s are invariant under the same cyclic permutations of $x_1, \ldots, x_n$ , [Gauss 1801], art. 360.III, [Gauss 1863], [Abel 1829], formulas (38)-(42), cf. [Neumann 2006], §§ 2, 4. In our subsection 2.3 we shall use this remark in order to complement Vandermonde’s most spectacular achievement.

For the actual calculation of the $V^{(i)}$’s Vandermonde proves the main theorem on symmetric polynomials and then is able to obtain the known solutions of the cubic and quartic equations anew, [Vandermonde 1774], §§
As to the general equations of degree $n > 4$ he does not arrive at any substantial steps towards an “algebraic” solution. For the composite degrees $n = 4, 6, 8, 9$ Vandermonde indicates modified “explicit” solutions in §§ XIII-XVII. In these cases there are indices $i$ having a common divisor $d > 1$ with $n$, say $n = n'd, i = i'd$. Then formally we can write $\sqrt[n']{(\Delta^{(i)} )^n} = \sqrt[n']{(\Delta^{(i')})^{n'}}$. Now the expression $\Delta^{(1)}$ looks simpler than, e. g., $\Delta^{(1)}$ insofar as it contains the $d$th powers of the $n$th roots of unity only, in other words the $n'$th roots of unity only. The $n$ roots $x_1, \ldots, x_n$ are arranged in $n'$ groups of $d$ summands each. These groups of $d$ summands have the form

$$x_k + x_{k+n'} + x_{k+2n'} + \cdots + x_{k+(d-1)n'} \quad (1 \leq k \leq n'). \quad (23)$$

Starting from $\Delta^{(2)}$ or $\Delta^{(3)}$ or $\Delta^{(4)}$, resp., Vandermonde succeeds to build up explicit solutions after permuting the $x_1, \ldots, x_n$ suitably in $\Delta^{(2)}$ or $\Delta^{(3)}$ or $\Delta^{(4)}$, resp. Certain sums of the special kind (23) occur three decades later in Gauss’s cyclotomy theory again and are baptized by Gauss as periods, [Gauss 1801], art. 343. This fact had led Lebesgue to the conjecture that for Gauss the concept of period could have been suggested by his early reading of Vandermonde, [Lebesgue 1940], pp. 33-34, 38. From a purely mathematical point of view this hypothesis looks at least well admissible but we reject it for various reasons which will be discussed in our next section on Gauss.

2.2. Quartic Equations and 5th Roots of Unity

As announced above for the quartic equation

$$(x - a)(x - b)(x - c)(x - d) = x^4 + N x^3 + P x^2 + Q x + R = 0$$

with the roots $a, b, c, d$ Vandermonde indicates the solutions in the modified and elegant form

$$x = \frac{1}{4} \left[ \epsilon_1 \sqrt{(a + b - c - d)^2} + \epsilon_2 \sqrt{(a + c - b - d)^2} \right.$$  
$$+ \epsilon_3 \sqrt{(a + d - b - c)^2} \left. + \frac{1}{4} (-N) \right] \quad (24)$$

with $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and $\epsilon_1 \epsilon_2 \epsilon_3 = +1$, [Vandermonde 1774], § XIII. The same formulas can be found with Lagrange, [Lagrange 1770-1771], § 32. Vandermonde leaves it at that although the conditions imposed on $\epsilon_1, \epsilon_2, \epsilon_3$
are still insufficient to select the admissible triples of square roots as realized by Lagrange. But the latter author shows the way out when we take into account that the quantity

$$Π := (a + b - c - d)(a + c - b - d)(a + d - b - c)$$

is invariant under all permutations of $a, b, c, d$. Hence $Π = −N^3 + 4NP − 8Q$ is known from the given equation. Therefore we should stipulate that

$$\sqrt{(a + b - c - d)^2} \cdot \sqrt{(a + c - b - d)^2} \cdot \sqrt{(a + d - b - c)^2} = Π.$$

On the other hand, the squares $(a + b - c - d)^2, (a + c - b - d)^2, (a + d - b - c)^2$ are only permuted with each other when we permute $a, b, c, d$ in all possible ways. Therefore those squares satisfy an equation of third degree the coefficients of which can be calculated by means of the given equation, [Vandermonde 1774], § XVI. We shall use this fact in our last section to construct the regular 17-gon by ruler and compass circumventing the primitive roots (mod 17) and closely following Vandermonde’s ideas.

Without loss of generality in the given equation we can assume $N = 0$. Then according to Eqn. (24) the solutions are sums of three square roots. This fact was as early as 1738 published by Euler who wrote

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

and deduced a cubic equation for $A, B, C$, [Euler 1738], § 5. Moreover, he derived the equality $\sqrt{A} \cdot \sqrt{B} \cdot \sqrt{C} = -Q/8$ which coincides with Lagrange’s condition (notice the factor $1/4$ in Eqn. (24)). As in Eqn. (24) Euler indicated the three remaining roots in the form

$$\sqrt{A} - \sqrt{B} - \sqrt{C}, \quad \sqrt{B} - \sqrt{A} - \sqrt{C}, \quad \sqrt{C} - \sqrt{A} - \sqrt{B}$$

which indeed means nothing but Eqn. (24).

Vandermonde uses his approach to quartic equations just described above in order to solve the 5th cyclotomic equation

$$\frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 = 0.$$

In this case he obtains a reducible auxiliary equation of degree 3 which eventually gives

$$x = \frac{1}{4} \left[ -1 + \epsilon_1 \sqrt{5} + \epsilon_2 \sqrt{-5 + 2\sqrt{5}} + \epsilon_3 \sqrt{-5 - 2\sqrt{5}} \right] \quad (25)$$

\[15\] Lagrange himself erroneously wrote $+N^3 - 4NP + 8Q$ which was realized by the editor J.-A. Serret.
with $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$ and $\epsilon_1 \epsilon_2 \epsilon_3 = +1$, [Vandermonde 1774], § XXIII. Here
the square roots should be chosen such that
\[
\sqrt{5} \cdot \sqrt{-5 + 2\sqrt{5}} \cdot \sqrt{-5 - 2\sqrt{5}} = -5.
\]
Of course, this result should coincide with (6). This follows indeed from
\[
\left[\sqrt{-5 + 2\sqrt{5}} \pm \sqrt{-5 - 2\sqrt{5}}\right]^2 = -10 \pm 2\sqrt{5}.
\]

2.3. 11th Roots of Unity

The unquestionable apex of Vandermonde’s theory is the representation
of the 11th roots of unity by means of radicals which has no counterpart in
Lagrange’s work at that time, see [Vandermonde 1774], § XXXV. Vander-
monde’s own exposition was very sketchy whereas some three decades later
Lagrange at length commented on Vandermonde’s solution of $x^{11} - 1 = 0$
and $x^5 - 1 = 0$, [Lagrange 1808], Note XIV. First of all, for the reciprocal
equation
\[
x^{10} = x^{10} + x^9 + \ldots + x + 1 = 0
\]
with the roots $r, r^2, \ldots, r^{10}$ Vandermonde calculates the auxiliary equation
\[
x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0
\]
with the five roots
\[
a := -r - r^{10}, \quad b := -r^2 - r^9, \quad c := -r^3 - r^8, \quad d := -r^4 - r^7, \quad e := -r^5 - r^6.
\]
Then he goes on to write down a series of intrinsic relations among those
roots:
\[
a^2 = -b + 2, \quad b^2 = -d + 2, \quad c^2 = -e + 2, \quad d^2 = -c + 2, \quad e^2 = -a + 2, \quad (26)
\]
\[
ab = -a - c, \quad bc = -a - e, \quad cd = -a - d, \quad de = -a - b, \quad (27)
\]
\[
ac = -b - d, \quad bd = -b - e, \quad ce = -b - c, \quad (28)
\]
\[
ad = -c - e, \quad be = -c - d, \quad ae = -d - e, \quad (29)
\]
\[
a + b + c + d + e - 1 = 0. \quad (31)
\]

Vandermonde’s further constructions are based on his fundamental ob-
observation that the relations (26)-(31) are only permuted with each other

\[\text{In recent terminology these relations entail that the subdomain } \mathbb{Z}[a, b, c, d, e] = \mathbb{Z}[a] = \mathbb{Z}[b] = \mathbb{Z}[c] = \mathbb{Z}[d] = \mathbb{Z}[e] \text{ of the number field } \mathbb{Q}(a, b, c, d, e) = \mathbb{Q}(a) = \mathbb{Q}(b) = \mathbb{Q}(c) = \mathbb{Q}(d) = \mathbb{Q}(e) \text{ is the (torsion-free, hence free) abelian group } \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot a + \mathbb{Z} \cdot b + \mathbb{Z} \cdot c + \mathbb{Z} \cdot d + \mathbb{Z} \cdot e \text{ with the relation (31). Its rank is } 5 \text{ which follows from the irreducibility of } x^{10} + x^9 + \ldots + x + 1 \text{ according to [Gauss 1801], art. 341.} \]
when one applies the cyclic permutation \((abdec) = \begin{pmatrix} a & b & c & d & e \\ b & d & e & c & a \end{pmatrix}\) and its powers to \(a, b, c, d, e\). To use his resolvents (19) he has to take into consideration the 5th roots of unity \(1, \rho, \rho^2, \rho^3, \rho^4\) where \(\rho\) denotes a primitive 5th root of unity. That is why he introduces the expressions

\[
\Theta^{(i)} := a + \rho^i \cdot b + \rho^{-i} \cdot d + \rho^{2i} \cdot c + \rho^{-2i} \cdot e
\]

\((1 \leq i \leq 4)\) which are multiplied by \(\rho^{-i}\) when one performs the cyclic permutation \((abdec)\). Vandermonde’s own notations are

\[
\Delta^{(1)} := \Theta^{(1)}, \quad \Delta^{(2)} := \Theta^{(4)}, \quad \Delta^{(3)} := \Theta^{(2)}, \quad \Delta^{(4)} := \Theta^{(3)}.
\]

Hence the quantities

\[
V^{(i)} := (\Delta^{(i)})^5
\]

are invariant under \((abdec)\) and Vandermonde concludes from his explicit calculations that all \(V^{(i)}\)’s are linear combinations of \(1, \rho, \rho^2, \rho^3, \rho^4\) with rational integer coefficients. Then the Eqns. (22) and (34) enable him to display \(a, b, c, d, e\) by means of iterated square roots and 5th roots.

Here we want to emphasize how important the relations (26) are and first we are going to prove the following assertion.

**Lemma.**

Let

\[
f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)
\]

be a polynomial such that there is a rational function \(\vartheta(X)\) with the property

\[
x_2 = \vartheta(x_1), \ldots, x_{i+1} = \vartheta(x_i), \ldots, x_1 = \vartheta(x_n).
\]

Let \(W(x_1, \ldots, x_n)\) be a rational function the coefficients of which are considered to be invariant under the cyclic permutation \((x_1 \ldots x_n)\). Suppose that, moreover, \(W\) is invariant under the cyclic permutation \((x_1 \ldots x_n)\).

Then \(W\) depends only on the elementary symmetric polynomials of \(x_1, \ldots, x_n\), i.e. on the coefficients of \(f(x)\).

\[\text{LOL-Ch16-P15 of 40}\]
Proof. Put \( W'(X) := W(X, \vartheta(X), \vartheta^2(X), \vartheta^3(X), \ldots, \vartheta^{n-1}(X)) \). Then we have \( W'(x_1) = W(x_1, \ldots, x_n) \). The invariance of \( W \) gives us \( W'(x_1) = W'(x_2) = \cdots = W'(x_n) \) and further

\[
W = \frac{1}{n} \cdot (W'(x_1) + \cdots + W'(x_n)).
\]

The right-hand side is symmetric in \( x_1, \ldots, x_n \). This means that \( W \) depends only on the elementary symmetric polynomials of \( x_1, \ldots, x_n \) which was to be proved. \( \Box \)

By the way, the proof is modeled on the arguments in Abel’s paper [Abel 1829]. In Vandermonde’s case the premises of the Lemma are satisfied by \( \vartheta(X) = -X^2 + 2 \) in view of (26). Now we can reproduce his results on the \( V^{(i)} \)'s and are additionally able to handle the relations between the radicals \( \sqrt[n]{V^{(i)}} \).

Relations (26)-(31) allow us to write any rational function \( W(a, b, c, d, e) \) in the form

\[
W = A \cdot a + B \cdot b + C \cdot c + D \cdot d + E \cdot e + F. \quad (35)
\]

For our numerical calculations the following corollary will be of some use.

**Corollary.**

If \( W = A \cdot a + B \cdot b + C \cdot c + D \cdot d + E \cdot e + F \) is invariant under the cyclic permutation \( (abdce) \) then

\[
W = \frac{1}{5} \cdot (A + B + C + D + E) + F. \quad (36)
\]

(See [Lebesgue 1940], pp. 35-36.)

**Proof.** Applying the permutation \( (abdce) \) and its powers to \( W \) we obtain the additional equalities

\[
W = A \cdot b + B \cdot d + C \cdot e + D \cdot c + E \cdot a + F \quad (37)
\]

\[
W = A \cdot d + B \cdot c + C \cdot a + D \cdot e + E \cdot b + F \quad (38)
\]

\[
W = A \cdot c + B \cdot e + C \cdot b + D \cdot a + E \cdot d + F \quad (39)
\]

\[
W = A \cdot e + B \cdot a + C \cdot d + D \cdot b + E \cdot c + F. \quad (40)
\]

Taking the average of the right-hand sides of (35), (37)-(40) we get

\[
W = \frac{1}{5} \cdot (A + B + C + D + E)(a + b + c + d + e) + F
\]

and with regard to (31) we have the formula (36). \( \Box \)

Of course, the explicit determination of the \( V^{(i)} \)'s requires a great amount of calculations. Vandermonde relies on his expansions of the \( V^{(i)} \)'s in sums of \( (abdce) \)-invariant terms exposed in § XXVIII of his treatise. Here we indicate the final results only:

\[
V^{(1)} = 196 + 130 \rho - 90 \rho^4 - 255 \rho^2 + 20 \rho^3 \quad (41)
\]
V(2) = 196 + 130\rho^4 - 90\rho - 255\rho^3 + 20\rho^2 \quad (42)
V(3) = 196 + 130\rho^2 - 90\rho^3 - 255\rho^4 + 20\rho \quad (43)
V(4) = 196 + 130\rho^3 - 90\rho^2 - 255\rho + 20\rho^4. \quad (44)

The quantities
\[ x_1, x_2, x_3, x_4, x_5 \]
are now expressions of the form
\[ x = \frac{1}{5} \left[ 1 + \sqrt[5]{V(1)} + \sqrt[5]{V(2)} + \sqrt[5]{V(3)} + \sqrt[5]{V(4)} \right]. \quad (45)\]

After inserting the explicit values of the 5th roots of unity Vandermonde obtains
\[ \sqrt[5]{V(1)} = \sqrt[5]{\frac{11}{4}} \left( 89 + 25\sqrt{5} - 5\sqrt{-5} + 2\sqrt{5} + 45\sqrt{-5} - 2\sqrt{5} \right) \quad (46)\]
\[ \sqrt[5]{V(2)} = \sqrt[5]{\frac{11}{4}} \left( 89 + 25\sqrt{5} + 5\sqrt{-5} + 2\sqrt{5} - 45\sqrt{-5} - 2\sqrt{5} \right) \quad (47)\]
\[ \sqrt[5]{V(3)} = \sqrt[5]{\frac{11}{4}} \left( 89 - 25\sqrt{5} - 5\sqrt{-5} + 2\sqrt{5} - 45\sqrt{-5} - 2\sqrt{5} \right) \quad (48)\]
\[ \sqrt[5]{V(4)} = \sqrt[5]{\frac{11}{4}} \left( 89 - 25\sqrt{5} + 5\sqrt{-5} + 2\sqrt{5} + 45\sqrt{-5} - 2\sqrt{5} \right). \quad (49)\]

He leaves it at that and does not take care of the ambiguities of the radicals. It is not difficult to calculate
\[ \Delta(1) \cdot \Delta(2) = \Delta(3) \cdot \Delta(4) = 11. \]

This means that the four quantities \( \Delta(1), \Delta(2), \Delta(3), \Delta(4) \) are complex numbers of modulus \( \sqrt{11} \) since \( a, b, c, d, e \) are real numbers and either of the two couples \( (\Delta(1), \Delta(2)), (\Delta(3), \Delta(4)) \) consists of numbers which are complex conjugate to each other. For the radicals we have the relations
\[ \sqrt[5]{V(1)} \cdot \sqrt[5]{V(2)} = \sqrt[5]{V(3)} \cdot \sqrt[5]{V(4)} = 11. \quad (50)\]

Moreover, we have further relations at our disposal since the products \( (\Delta(1))^2 \cdot \Delta(4) \) and \( (\Delta(1))^3 \cdot \Delta(3) \) turn out to be \( (abdce) \)-invariant as well. For our purposes it will be sufficient to calculate \( (\Delta(1))^2 \cdot \Delta(4) \) since we know already \( (\Delta(1))^5 \). It is not difficult to expand our product of three factors in a sum of 7 \( (abdce) \)-invariant terms following an idea of Vandermonde at that. We skip the details of the calculations and indicate the final result:
\[ (\Delta(1))^2 \cdot \Delta(4) = S(a^3) + (\rho^2 + 2\rho^4)S(ab^2) + (\rho^4 + 2\rho^3)S(ad^2) + 
+ (\rho + 2\rho^2)S(ac^2) + (\rho^3 + 2\rho)S(ae^2) 
+ (2 + 2\rho^2 + 2\rho^3)S(abd) + (2 + 2\rho + 2\rho^4)S(abc)
= 11(-2\rho + 2\rho^2 + \rho^3). \]
In these equations the symbols $S(\ldots)$ denote the sums taken over all expressions which result from the argument after applying the cyclic permutation $(abdce)$ and its powers to that argument.\footnote{The equality $(-2\rho + 2\rho^2 + \rho^3)(-2\rho^4 + 2\rho^3 + \rho^2) = 11$ shows us that the complex number $(\Delta^{(1)})^2 \cdot \Delta^{(4)}$ has the correct modulus $(\sqrt{11})^3$. The factorization of 11 can be refined such that one obtains the unique prime decomposition of 11 in the euclidean domain $\mathbb{Z}[\rho]$: 

$$11 = (\rho^2 + \rho^3 - \rho^4)(1 + \rho^2 - \rho^3)(\rho^3 + \rho^2 - \rho)(1 + \rho^3 - \rho^2).$$}

For the radicals in Eqn. (45) we obtain the relation

$$\left(\sqrt[5]{V^{(1)}}\right)^2 \cdot \sqrt[5]{V^{(4)}} = 11(-2\rho + 2\rho^2 + \rho^3).$$

The equations (50) and (51) taken together show us that all radicals $\sqrt[5]{V^{(i)}}$ are uniquely determined by the value of $\sqrt[5]{V^{(1)}}$. Hence the formula (45) yields five values of $x$ only as it should be expected.

2.4. “Vandermonde’s Condition”

In § VI, XXXVI Vandermonde affirms to the reader that for primes $n = 2m + 1$ the auxiliary equation of degree $m$ associated with $x^n - 1 = 0$ “can always be solved easily”. It seems he formed this opinion on the examples for $n \leq 11$. Was Vandermonde really right? How could he himself have generalized his method from $n = 11$ to other primes as well? We are going to expose his basic ideas in as much generality as is possible without abandoning the framework of his treatise.

Let $n = 2m + 1$ be an odd prime and $r$ be a primitive $n$th root of unity. We define $m$ quantities in an upper numbering as follows:

$$x^{(k)} := -\rho^k - \rho^{-k} \quad (1 \leq k \leq m)$$

and some of them in a lower numbering:

$$x_i := -\rho^{(2^i)} - \rho^{-(2^i)} \quad (0 \leq i).$$

Now it is evident that Vandermonde’s method hinges on the relations (26). In our setting we have $x_i^2 = -x_{i+1} + 2$. From there we conclude that Vandermonde’s method will work if and only if the $x_i$ will run through the whole set $\{x^{(1)}, \ldots, x^{(m)}\}$. This can happen if and only if there are precisely $m$ unordered pairs $\{2^i \mod n, -2^i \mod n\}$. The latter condition is in turn equivalent to the condition that $m$ be the least positive exponent $k$ with $2^k \equiv \pm 1 \pmod{n}$ or, in other words, $2^m \equiv \pm 1 \pmod{n}$ and $2^k \not\equiv \pm 1 \pmod{n}$.\footnote{The equality $(-2\rho + 2\rho^2 + \rho^3)(-2\rho^4 + 2\rho^3 + \rho^2) = 11$ shows us that the complex number $(\Delta^{(1)})^2 \cdot \Delta^{(4)}$ has the correct modulus $(\sqrt{11})^3$. The factorization of 11 can be refined such that one obtains the unique prime decomposition of 11 in the euclidean domain $\mathbb{Z}[\rho]$: 

$$11 = (\rho^2 + \rho^3 - \rho^4)(1 + \rho^2 - \rho^3)(\rho^3 + \rho^2 - \rho)(1 + \rho^3 - \rho^2).$$}
(mod $n$) for $0 < k < m$. For sake of brevity we shall call this “Vandermonde’s condition”.

The special prime numbers $n = 2m + 1$ such that $m$ is also a prime number satisfy this condition. Indeed for the prime number 5 we have $2^2 \equiv -1 \pmod{5}$. Further for primes $n = 2m + 1 > 5$ the primes $m$ are odd. In this case we can prove even more. For every prime $g \equiv 1 \pmod{n}$ is excluded, thus $g \pmod{n}$ has order $m$ or $2m$. If $g^m \equiv 1 \pmod{n}$ then $(-g)^m = -g^m \equiv -1 \pmod{n}$ and $(-g) \pmod{n}$ is a primitive root mod $n$. Indeed, $g^2 \equiv 1 \pmod{n}$ is odd. In particular, if 2 is a primitive root mod $n$ then we have $2^m \equiv -1 \pmod{n}$, and a congruence $2^k \equiv \pm 1 \pmod{n}$ with $0 < k < m$ is excluded. On the other hand, if $(-2)$ is a primitive root mod $n$ then we have $(-2)^m \equiv -1 \pmod{n}$, hence $2^m \equiv 1 \pmod{n}$, and a congruence $2^k \equiv \pm 1 \pmod{n}$ with $0 < k < m$ is impossible since $m$ is odd.

Thus the class of primes just discussed contains 25 primes < 1000, namely


Besides we have 75 further primes < 1000


for which Vandermonde’s condition can be checked immediately. Among the primes < 100 only the numbers 17, 31, 41, 43, 73, 89, 97 do not satisfy Vandermonde’s condition. In particular, 17 is missing there. Nevertheless one can construct the regular 17-gon by ruler and compass following Vandermonde’s ideas in a slightly modified way as we will show in our last section. A further remark refers to the so-called Fermat primes of the form $F_k = 2^{2^k} + 1$ like 3, 5, 17, 257 and 65 537 (further instances are not

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19 In his paper [Loewy 1918], p. 192, Loewy overlooked this fact and referred unnecessarily to [Abel 1829], § 3, theorem, in order to underpin his arguments.

20 This sequence of primes was very quickly computed with the help of the tables [Jacobi 1956]. For every prime $n < 1000$ these tables contain the least positive primitive root mod $n g$, the map $i \mapsto N \equiv g^i \pmod{n}$ (table of “numeri”), the inverse map $N \mapsto ind(N)$ (table of “indices”) and two further tables, $ind(x) \mapsto ind(x+1)$ and $ind(x) \mapsto ind(x-1)$. 

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known until now). For \( k > 1 \) we have \( 2^{(2k)} = (-2)^{(2k)} \equiv -1 \pmod{F_k} \) where the exponent \( 2^k \) is less than \( 1/2 \cdot (F_k - 1) = 2^{(2k-1)} \). In other words, for a Fermat prime with \( k \geq 2 \) Vandermonde’s method would never work immediately.

2.5. “Pre-Gaussian” Cyclotomy Theory

Despite of all limitations in Vandermonde’s work we feel entitled to agree with Alfred Loewy (1873-1935) who said that

“Vandermonde was the first who saw and carried out the right method to solve the equation \( x^{2m+1} = 1 \) for prime numbers \( 2m + 1 \)” [Loewy 1918], p. 194. 21

Let \( n = 2m + 1 \) be a prime without further restrictions for the time being and \( r \) a primitive \( n \)th root of unity. Here we are going to develop a fragment of cyclotomy theory entirely based on the ideas and tools of Euler, Vandermonde and Lagrange. In short we will expose a kind of “pre-Gaussian” cyclotomy theory. Throughout we will abstain from the conscious use of primitive roots mod \( n \) for arbitrary primes \( n \), in other words, from the use of a powerful tool due only to Gauss.

With the notations just introduced above the \( m \) quantities \( x^{(1)} = -r - r^{-1}, \ldots, x^{(m)} = -r^m - r^{-m} \) in the upper numbering satisfy an equation

\[
F(x) = x^m - x^{m-1} - (m-1)x^{m-2} + (m-2)x^{m-3} + \frac{(m-2)(m-3)}{1 \cdot 2}x^{m-4} - \ldots = 0,
\]

[Vandermonde 1774], § VI. All \( x^{(i)} \)'s are power sums of \( r, r^{-1} \), therefore polynomials of each other with rational integer coefficients since all \( n \)th roots of unity \( r, r^2, \ldots, r^{2m} \neq 1 \) are primitive and can replace each other. The \( n \)th roots of unity \( r, r^{-1}, r^2, r^{-2}, \ldots, r^m, r^{-m} \) in this order just are the roots of \( m \) quadratic equations

\[
y^2 + x^{(i)} \cdot y + 1 = 0 \quad (1 \leq i \leq m),
\]

this means

\[
(x^{(i)})^2 - 4 = (r^i - r^{-i})^2, \quad y = \frac{-x^{(i)} \pm \sqrt{(x^{(i)})^2 - 4}}{2} = r^i \text{ or } r^{-i}.
\]

21 Vandermonde hat demnach, wie man wohl sagen kann, als erster die richtige Meth-ode zur Auflösung der Gleichung \( x^{2m+1} = 1 \) für primzahliges \( 2m + 1 \) erkannt und durchgeführt.
Conversely, assume for the moment that only the equations (52) and (53) would be given. Denote for any $i$ the solutions of (53) by $r, r^{-1}$. Then $x^{(i)} = -r - r^{-1}$. Inserting this equality in Eqn. (52) we will get

$$\frac{r^n - 1}{r - 1} = r^{n-1} + r^{n-2} + \ldots + r + 1 = 0.$$ 

This means that $r, r^{-1}$ really are $n$th roots of unity and the $m$ disjoint pairs $\{r^k, r^{-k}\}$ exhaust all $n$th roots of unity $\neq 1$.

Moreover we observe that for any pair $(i, j)$ of indices the quotient

$$\frac{r^i - r^{-i}}{r^j - r^{-j}} = \frac{(r^i - r^{-i})(r^j - r^{-j})}{(r^j - r^{-j})^2} = \frac{(r^i - r^{-i})(r^j - r^{-j})}{(x(j))^2 - 4}$$

is symmetric in $r, r^{-1}$, hence a rational function of $r + r^{-1} = -x^{(1)}$ with rational coefficients. This means that any two square roots $\sqrt{(x^{(i)})^2 - 4}$, $\sqrt{(x^{(j)})^2 - 4}$ differ from each other only by a rational function of $x^{(1)}$ with rational coefficients. In particular, every $n$th root of unity is a rational function (with rational coefficients) of $x^{(1)}$ and a single quadratic radical $\sqrt{(x^{(i)})^2 - 4}$ where $i$ can be chosen arbitrarily. Anticipating some further considerations, we inform the reader that below, for odd $m$, this result will be reinforced considerably; insofar as in this case the radicals $\sqrt{(x^{(i)})^2 - 4}$ can be replaced by $\sqrt{-n}$. This explains Euler’s successful treatment of the 7th roots of unity.

Euler’s equation (11) can also be generalized to the equation satisfied by the $2m$ quantities $\pm \sqrt{(x^{(i)})^2 - 4}$. The polynomial

$$G(x) := \prod_{i=1}^{m} \left( x - \sqrt{(x^{(i)})^2 - 4} \right) \left( x + \sqrt{(x^{(i)})^2 - 4} \right) = \prod_{i=1}^{m} (x^2 - (x^{(i)})^2 + 4) \hspace{1cm} (55)$$

has coefficients which are symmetric in $x^{(1)}, \ldots, x^{(m)}$ and, therefore, rational integers. Thus $G(x)$ shows the desired properties.

Furthermore we have

$$(r^i - r^{-i})^2 = -(1 - r^{2i})(1 - r^{-2i}) \hspace{1cm} (56)$$

for all exponents $i$. Multiplying all these $m$ equations we obtain

$$\left( \prod_{i=1}^{m} (r^i - r^{-i}) \right)^2 = (-1)^m \prod_{i=1}^{m} (1 - r^{2i})(1 - r^{-2i}) = (-1)^m \cdot (x^{2m} + x^{2m-1} + \ldots + x + 1)|_{x=-1} = (-1)^m \cdot n \hspace{1cm} (57)$$
since $r^2, r^{-2}, r^4, r^{-4}, \ldots, r^{2m}, r^{-2m}$ are all $n$th roots of unity $\neq 1$. Taking the square root and using Eqns. (54) we see that

$$\prod_{i=1}^{m} \sqrt{(x^{(i)})^2 - 4} = \pm \prod_{i=1}^{m} (r^i - r^{-i}) = \pm \sqrt{(-1)^m \cdot n}.$$  \hspace{3cm} (58)

The last equality shows us that the square root $\sqrt{(-1)^m \cdot n}$ is a linear combination of the $n$th roots of unity with rational integer coefficients. This remarkable (and momentous) fact could very well have been proved by Euler, Lagrange or Vandermonde but it wasn’t. The case $n = 3$ is obvious whereas the cases $n = 5, 7$ are settled implicitly in the calculations of the $n$th roots of unity with Euler, Lagrange and Vandermonde. The proof in the general case is due to Gauss (whose proof is different from the one given here), [Gauss 1801], art. 356.

A closer inspection exhibits that the product $\prod_{i=1}^{m} (r^i - r^{-i})$ has the form of a polynomial $H(r, r^{-1})$ in $r, r^{-1}$ with rational integer coefficients and the property $H(r^{-1}, r) = (-1)^m \cdot H(r, r^{-1})$. From there it follows that for even $m$ $H(r, r^{-1})$ is symmetric in $r, r^{-1}$, hence a polynomial in $r + r^{-1} = -x^{(1)}$ and $r \cdot r^{-1} = 1$ with rational integer coefficients. In other words, for even $m$ the quadratic radical $\sqrt{(-1)^m \cdot n} = \sqrt{n}$ is already a polynomial in $x^{(1)}$ or in any other $x^{(i)}$. For odd $m$ we have $H(r^{-1}, r) = -H(r, r^{-1})$ and $H(r, r^{-1})$ takes the special form

$$\pm \sqrt{n} = H(r, r^{-1}) = (r - r^{-1}) \cdot S(r, r^{-1}) = (r - r^{-1}) \cdot P(x^{(1)}) = \pm \sqrt{(x^{(1)})^2 - 4} \cdot P(x^{(1)})$$

with a symmetric polynomial $S$ and some polynomial $P$. From there we deduce that the two square roots $\sqrt{n}$ and $\sqrt{(x^{(1)})^2 - 4}$ differ from each other only by a rational function of $x^{(1)}$. In summary, for odd $m$ the $n$th roots of unity are rational functions of $x^{(1)}$ (or of any other $x^{(i)}$) and $\sqrt{n}$ with rational coefficients. In a more explicit manner we can write

$$r^i - r^{-i} = \pm \frac{\sqrt{-n}}{\prod_{j \neq i} (r^j - r^{-j})}.$$  \hspace{3cm}

The numerator of this quotient is symmetric in $r, r^{-1}$, therefore a polynomial in $x^{(1)}$. Together with the definition $x^{(i)} := -r^i - r^{-i}$ this gives us the desired result since $x^{(i)}$ is a polynomial in $x^{(1)}$.

2.6. “Vandermonde’s Condition” Again

Now in order to treat $x^n - 1 = 0$ or, more precisely, the equation (52) using Vandermonde’s ideas one has to impose on $n$ the restriction that
$2^m \equiv \pm 1 \pmod{n}$ and $2^k \not\equiv \pm 1 \pmod{n}$ for $0 < k < m$. The series of the primes in question contains 100 primes $< 1000$ and begins as follows:

3, 5, 7, 11, 13, 19, 23, 29, 37, 47, 53, 59, 61, 67, 71, 79, 83, ...

Our aim is to solve Eqn. (52) by $m$th roots of unity and radicals of index $m$. Under Vandermonde’s condition we can switch to the lower numbering and on the analogy of (26)-(31) we have the “cyclic” relations

$$x_i^2 = -x_{i+1} + 2 \quad (1 \leq i \leq m-1), \quad x_m^2 = -x_1 + 2$$

as well as $m(m-1)/2$ formulas for the products $x_i \cdot x_j (i < j)$. All these relations are only permuted with each other when one performs the cyclic permutation $(x_1 \ldots x_m)$. Let $\rho$ be a primitive $m$th root of unity. Then one has to form the expressions

$$\Delta(i) = x_1 + \rho^i \cdot x_2 + \cdots + \rho^{(m-1)i} \cdot x_m \quad (1 \leq i \leq m-1)$$

and their $m$th powers $V(i)$ which turn out to be invariant under the permutation $(x_1 \ldots x_m)$. Now we can apply the Lemma and see the $V(i)$’s to be linear combinations of the $m$th roots of unity with rational integer coefficients. Unfortunately, Vandermonde himself proves this fact only for $n = 5, 11$ in § XXIII, XXXV. That is why here we have interpolated the Lemma in our comment on Vandermonde’s text.

All products $\Delta(i) \cdot \Delta^{(m-i)}$ are also invariant under the permutation $(x_1 \ldots x_m)$, and it is not difficult to expand them in sums of $(x_1 \ldots x_m)$-invariant terms which can be calculated separately.

$$\Delta(i) \cdot \Delta^{(m-i)} = (x_1^2 + \cdots + x_m^2)$$

$$+ \sum_{j=1}^{m-1} (\rho^{ji} + \rho^{-ji}) \cdot (x_1 x_{j+1} + x_2 x_{j+2} + \cdots + x_m x_j)$$

$$+ \frac{1}{2} (1 + (-1)^m) (-1)^i (x_1 x_2 + \cdots)$$

$$= (-1 + 2m) + (-1)(-2) = n.$$

These equalities show us that all resolvents $\Delta(i)$ are $\neq 0$. More precisely, $\Delta^{(m-i)}$ is the complex conjugate of $\Delta(i)$ since $x_1, \ldots, x_m$ are real numbers, hence $\Delta(i)$ is a complex number of modulus $\sqrt{n}$.

On the analogy of (45) we obtain “explicit” solutions of (26). First of all, with regard to $x_{i+1} = -x_i^2 + 2$ we see that

$$x_{i+1}^2 - 4 = x_i^2 \cdot (x_i^2 - 4), \quad \left(\frac{-x_i \pm \sqrt{x_i^2 - 4}}{2}\right)^2 = \frac{-x_{i+1} \pm \sqrt{x_{i+1}^2 - 4}}{2}.$$
Therefore, beginning with the solutions \( r, r^{-1} \) of \( y^2 + x_1 \cdot y + 1 = 0 \) we can arrange all solutions of the quadratic equations in the sequence \( r, r^{-1}, r^2, r^{-2}, r^4, r^{-4}, \ldots \).

The polynomial (55) can be specified very easily using Eqn. (52). With respect to the quantities \( \pm \sqrt{x_1^2 - 4} = \pm \sqrt{-x_{i+1} - 2} \) we put \( x^2 := -x_{i+1} - 2 \) and form the polynomial

\[
(x^2 + x_1 + 2)(x^2 + x_2 + 2) \cdots (x^2 + x_m + 2) =: G(x)
\]

which can be derived immediately from (52):

\[
G(x) = (-1)^m \cdot F(-x^2 - 2) = (-1)^m \cdot F(-x^2 - 2).
\]

Here \( F(x) \) denotes the polynomial in (52) with the zeroes \( x_1, \ldots, x_m \).

### 2.7. 7th Roots of Unity

To round off this section on Vandermonde we would like to treat the equation \( x^n - 1 = 0 \) for \( n = 7 \) following Vandermonde’s ideas whereas the cases \( n = 3, 5 \) are left to the reader. With a primitive 7th root of unity \( r \) and a primitive 3rd root of unity \( \rho = \frac{-1 + \sqrt{-3}}{2} \) we define

\[
x_1 = -r - r^6, \quad x_2 = -r^2 - r^5, \quad x_3 = -r^4 - r^3,
\]

\[
\Delta^{(1)} = x_1 + \rho \cdot x_2 + \rho^2 \cdot x_3, \quad \Delta^{(2)} = x_1 + \rho^2 \cdot x_2 + \rho \cdot x_3.
\]

In our case Eqn. (52) takes the form

\[
F(x) := x^3 - x^2 - 2x + 1 = 0
\]

which of course is nothing but Euler’s equation (8). The quantity \( \Delta^{(1)} \cdot \Delta^{(2)} \) is a symmetric polynomial in \( x_1, x_2, x_3 \):

\[
\Delta^{(1)} \cdot \Delta^{(2)} = x_1^2 + x_2^2 + x_3^2 - x_1 \cdot x_2 - x_2 \cdot x_3 - x_3 \cdot x_1 = 7.
\]

In § III Vandermonde indicates the expansion of \( (\Delta^{(1)})^3 \) in \( (x_1 x_2 x_3) \)-invariant components which reads as follows:

\[
(\Delta^{(1)})^3 = (x_1^3 + x_2^3 + x_3^3) + 3\rho(x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1) +
\]

\[
+ 3\rho^2(x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2) + 6x_1 x_2 x_3.
\]

Now by means of the specific relations

\[
x_1 x_2 = -x_1 - x_3, \quad x_2 x_3 = -x_2 - x_1, \quad x_3 x_1 = -x_3 - x_2
\]

one calculates each component of \( (\Delta^{(1)})^3 \) and \( (\Delta^{(2)})^3 \), resp., rather easily and obtains

\[
(\Delta^{(1)})^3 = (-7) \cdot \frac{1 + 3\sqrt{-3}}{2}, \quad (\Delta^{(2)})^3 = (-7) \cdot \frac{1 - 3\sqrt{-3}}{2}.
\]
This way Euler’s solutions of (8) occur here again. In the next step Euler’s equation (11) satisfied by $\pm \sqrt{-x_1-2}, \pm \sqrt{-x_2-2}, \pm \sqrt{-x_3-2}$ shows up when we use Eqn. (59):

$$0 = G(x) = -F(-x^2 - 2) = x^6 + 7x^4 + 14x^2 + 7.$$  

Furthermore, our previous general considerations show that the 7th roots of unity are rational functions of $\sqrt{-7}, \sqrt{-3}$ (via the 3rd roots of unity) and a single cubic radical. Thus the results of Euler’s calculations are confirmed.

We see that the approach to $x^7 - 1 = 0$ à la Vandermonde is at least conceptually superior to Euler’s due to the use of the relations (60).

It is to be regretted that Vandermonde did not consider the case $n = 13$ in detail. In this case he could have observed the phenomenon of “periods of 4 or 6 terms” not yet occurring in the former cases $n = 7, 11$. Let $r$ be a primitive 13th root of unity and $\rho = \frac{1 + \sqrt{-3}}{2}$ be a primitive 6th root of unity. In our previous notations we obtain six resolvents $\Delta^{(1)}, \ldots, \Delta^{(6)}$ among which there are the quantities

$$\Delta^{(2)} = (x_1 + x_4) + \rho^2(x_2 + x_5) + \rho^3(x_3 + x_6),$$
$$\Delta^{(4)} = (x_1 + x_4) + \rho^4(x_2 + x_5) + \rho^2(x_3 + x_6)$$

and

$$\Delta^{(3)} = (x_1 + x_3 + x_5) + \rho^3(x_2 + x_4 + x_6).$$

In these expressions the terms are summed according to the powers of $\rho$, thus new combinations of the 13th roots of unity show up like the “periods of 4 terms”

$$x_1 + x_4 = r^{(2^0)} + r^{-(2^6)} + r^{(2^3)} + r^{-(2^3)},$$
$$x_2 + x_5 = r^{(2^1)} + r^{-(2^1)} + r^{(2^4)} + r^{-(2^4)},$$
$$x_3 + x_6 = r^{(2^2)} + r^{-(2^2)} + r^{(2^5)} + r^{-(2^5)},$$

and the “periods of 6 terms”

$$x_1 + x_3 + x_5 = r^{(2^0)} + r^{-(2^6)} + r^{(2^2)} + r^{-(2^2)} + r^{(2^4)} + r^{-(2^4)},$$
$$x_2 + x_4 + x_6 = r^{(2^1)} + r^{-(2^1)} + r^{(2^3)} + r^{-(2^3)} + r^{(2^5)} + r^{-(2^5)}.$$  

The reader should notice that here in each sum the total number of terms is counted as the number of roots of unity involved in the sum but not with respect to the pairs $(r^i, r^{-i})$.

### 3. Gauss

How did Gauss come to study the cyclotomy? Could he have been influenced by Vandermonde? Before discussing these questions we are going
to look at his biography and the mathematical writings to which he had access in his early years. Gauss spent his childhood and early youth in Braunschweig (Brunswick) where in 1792 at age of 15 he entered the Collegium Carolinum which was a semi-academic science-oriented institution preparing young men for a career as well-qualified loyal bureaucrats and military personnel. The library of the Carolinum was unusually good and gave Gauss access to many of the best and most advanced textbooks in mathematics and the sciences and to classics like Isaac Newton’s (1643-1727) writings and John Wallis’s (1616-1703) “A Treatise of Algebra”. But it should be stressed that rather recent investigations into the extant catalogues of the Carolinum library have shown this library did not have writings of Pierre de Fermat (1601/1607? - 1665), Euler, Waring, Vandermonde, Lagrange and Adrien-Marie Legendre (1752-1833), let alone the publications of the learned academies in Berlin, Paris, St. Petersburg and London, [Küssner 1979], pp. 32-40. In October 1795 Gauss moved to Göttingen and enrolled as a student of classical philology and mathematics at the university. Gauss was a zealous user of the excellent Göttingen library which was one of the best all over Europe in its time. At last he could study the masters like Euler and Lagrange and their treatises on mathematics and the sciences in the publications of the European academies. G. Waldo Dunnington (1906-1974) compiled an impressive record of the books which Gauss had borrowed from the library, though in this record the summer semester 1796 is missing, [Dunnington 2004], pp. 398-404. Little is known about Gauss’s first semester but beginning with March 30, 1796, we have his invaluable mathematical diary (Notizenjournal) where a great many of entries refer to number theory and algebra, [Gauss 1796-1814].

3.1. The 17-Gon

It is well-known and often quoted that Gauss’s first entry in his diary reads as follows:

“The principles upon which the division of the circle depends, and geometrical divisibility of the same into seventeen parts, etc.”, [Gauss 1796-1814].

After almost three weeks, on April 18, 1796, he had written a short announcement of his discoveries in cyclotomy whereas this communication was published under the date of June 1, 1796, [Gauss 1863-1933], vol. I, p. 3. Gauss emphasized the then completely unexpected constructibility of

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22English translation quoted after [Dunnington 2004], p. 469. Original in Latin: Principia quibus innititur sectio circuli, ac divisibilitas eiusdem geometrica in septemdecim partes etc.,
the regular 17-gon and some other polygons by ruler and compass. Moreover, he wrote that these results are only “a corollary of a theory which is not yet complete”. The “complete theory” was eventually exposed in his first major opus *Disquisitiones arithmeticae* (1801), more precisely, in the seventh section of this work, [Gauss 1801]. Unfortunately, little is known about when, how and why Gauss began to study the problems of cyclotomy. He was hardly led to his theory by geometrical problems, probably algebraic and arithmetical questions got the theory going, cf. [Bachmann 1911], pp. 32-40. This opinion is backed by Gauss’s letter to the mathematician, physicist and astronomer Christian Ludwig Gerling (1788-1864) from January 6, 1819, where he said that as early as in his first semester in Göttingen he had obtained an important result in cyclotomy before he discovered the constructibility of the 17-gon, [Gauss 1863-1933], vol. X/1, p. 125. The result in question concerns the equation $x^n - 1 = 0$ for prime numbers $n = 2m + 1$. Gauss had seen that it would be appropriate to subdivide the $n$th roots of unity ($\neq 1$) $r, r^2, \ldots, r^{n-1}$ into two groups depending on whether the exponent with $r$ be a quadratic residue mod $n$ or a quadratic non-residue mod $n$. The sum over either group is nothing but a period of $m$ terms in his later terminology. These sums

$$\tau_1 = \sum_{i=1}^{m} r^{i^2}, \quad \tau_2 = \sum_{i=1}^{m} r^{h \cdot i^2}$$

(61)

where $h$ denotes a quadratic non-residue mod $n$ were thoroughly studied by Gauss, and they turned out to be quadratic irrationalities. Especially, he found the quadratic equation

$$x^2 + x - (-1)^m \frac{m}{2} = 0$$

satisfied by the two sums. Moreover, he could prove that

$$\frac{4(x^n - 1)}{x - 1} = Y(x)^2 - (-1)^m nZ(x)^2$$

for some polynomials $Y, Z$ with rational integer coefficients. To these results he alluded in his letter to Gerling, and this means he had discovered no less than the close and extremely important connections between cyclotomy, quadratic irrationalities and quadratic residues! These facts fit very well with Gauss’s studies on quadratic residues at that time and his efforts to prove the reciprocity law of quadratic residues. Eventually, his second entry in the diary on April 8, 1796, testified the first complete proof of that fundamental law. A systematic exposition of his first results in cyclotomy was given by Gauss in 1801, as he indicated to Gerling, [Gauss 1801], art. 124, 356, 357.
Now one could ask, how did Gauss then proceed to discover the constructibility of the 17-gon? In the case of \( x^{17} - 1 = 0 \) we have the two sums \( \tau_1, \tau_2 \) of 8 terms each according to Eqn. (61). We share Paul Bachmann’s (1837-1920) opinion that most probably in a flash of genius Gauss had seen how further to subdivide the sums of 8 terms each into 2 suitable sums of 4 terms each, further into sums of 2 terms each and to end up with the 17th roots of unity, [Bachmann 1911], p. 40. The principle of the iterated subdivisions is of arithmetical nature, and that is what Gauss had called “the interconnection of all roots on arithmetical grounds” (Zusammenhang aller Wurzeln unter einander nach arithmetischen Gründen) in his letter to Gerling, see also [Reich 2003]. The basic fact which Gauss had used was the existence of primitive roots mod \( n \) for any prime number \( n \), i.e. the existence of such residue classes \( g \mod n \) that the powers \( 1, g, g^2, g^3, \ldots, g^{n-2} \) run through all residue classes mod \( n \). This fact was first formulated by Euler and Johann Heinrich Lambert (1728-1777) but its first complete proof is due to Gauss, [Gauss 1801], art. 55-56 (with comments on Euler and Lambert). Those primitive roots mod \( n \) allowed Gauss to order the \( n \)th roots of unity in the series \( r, r^g, r(g^2), r(g^3), \ldots, r(g^{n-2}) \) where each member just is the \( g \)th power of the preceding one. It is this series which gives us the “right” order of the roots of unity for any prime number! For \( n = 17 \) one can choose \( g = 3 \mod 17 \) whereas Vandermonde’s methods do not work because \( 2^4 \equiv -1 \mod 17 \). In the general case let \( n - 1 = e \cdot f \) be a factorization. A period of \( f \) terms is determined by a geometric sub-progression of \( 1, g, g^2, g^3, \ldots, g^{n-2} \mod n \) with quotient \( g^e \mod n \). This means a period of \( f \) terms is a sum

\[
\eta_h = r^h + r^{hg^e} + r^{hg^2e} + \cdots + r^{hg^{(f-1)e}}
\]

(62)

where \( h \) denotes an arbitrary exponent \( \not\equiv 0 \mod n \). There is every reason to believe that Gauss first developed the theory of periods and only after that the solution of \( x^n - 1 = 0 \) by means of Lagrange (-Vandermonde) resolvents. One should notice that only half of a year later, on September 17, 1796, Gauss made a note in his diary of the expressions coincident with the Lagrange (-Vandermonde) resolvents. Like Vandermonde he hoped then for “a new method by means of which it will be possible to investigate, and perhaps try to invent, the universal solution of equations.” 23 The application to the cyclotomic equations is mentioned in January 1797 and July 1797.

23 English translation quoted after [Dunnington 2004], p. 472. Complete original in Latin. Nova methodus qua resolutionem aequationum universalem investigare forsitanque invenire licebit. Scilicet transm[utem] aequationem in aliam, cuius radices \( \alpha p^1 + \beta p^\gamma + \cdots \), ubi \( \sqrt[n]{1} = \alpha, \beta, \gamma \) etc. et \( n \) numerus aequationis gradum denotans.
We have no evidence whatsoever that Vandermonde could have known or used the existence of primitive roots mod \(n\) in the general case. At least this marks the decisive difference between him and Gauss. Even in the simplest cases \(n = 5, 11\) Vandermonde did not mention nor use that 2 is a primitive root mod \(n\). Nevertheless Lebesgue made the assertion that Gauss followed Vandermonde “step by step” in his exposition of the cyclotomy theory but there “he perfected Vandermonde very much”. For instance, as to the periods, Lebesgue says that “the method is that of Vandermonde, the results are those of Gauss”, [Lebesgue 1940], p. 38. We maintain that this judgement goes too far since Vandermonde like Euler confined himself to the sums \(r + r^{-1}\), i.e. to periods of 2 terms in Gauss’s sense, the introduction of which is clearly suggested by the reciprocal equations without any further sophistication.

Let us return for a while to the regular 17-gon as promised above. Lebesgue claimed that Vandermonde “had not understood the full importance of his method”, and he attempted to convince the reader that the constructibility of the 17-gon could have been derived rather transparently from Vandermonde’s method [Lebesgue 1940], p. 42. For “that method would have given Vandermonde the roots of \(x^{17} - 1 = 0\) by means of radicals of index 16, therefore, by means of superposition of square roots”. Apparently, here Lebesgue alluded to Eqn. (22) applied to \(\frac{x^{17} - 1}{x - 1} = x^{16} + x^{15} + \cdots + x + 1 = 0\), in other words to a representation of the 17th roots of unity by sums of radicals \(\sqrt[r]{V(i)}\). But the formation of the \(V(i)\)'s is inseparably tied up to a suitable order of the 17th roots of unity \(\neq 1\). Otherwise one could not prove that the \(V(i)\)'s would actually be linear combinations of the 16th roots of unity with rational coefficients. Hence one is in urgent need of a primitive root mod 17 which cannot be found with Vandermonde.

This situation raises a question: are there ways different from Lebesgue’s which do prove the constructibility of the regular 17-gon à la Vandermonde? Our answer will be affirmative, and we are going to pursue such a way. Of course Vandermonde’s Mémoire suggests to us to start with Eqn. (52) of degree 8

\[
x^8 - x^7 - 7x^6 + 6x^5 + 15x^4 - 10x^3 - 10x^2 + 4x + 1 = 0
\]

which has the roots \(-r^i - r^{-i}, (1 \leq i \leq 8)\), for a primitive 17th root of unity \(r\). We shall use a suitable lower numbering of these roots which looks as follows:

\[
\begin{align*}
x_1 &= -r - r^{-1}, & x_2 &= -r^2 - r^{-2}, \\
x_3 &= -r^4 - r^{-4}, & x_4 &= -r^8 - r^{-8},
\end{align*}
\]
\[ x_5 = -r^6 - r^{-6}, \quad x_6 = -r^{12} - r^{-12}, \]
\[ x_7 = -r^7 - r^{-7}, \quad x_8 = -r^3 - r^{-3}. \]  
(65)

For the first line we obtain
\[ x_{i+1} = -x_i^2 + 2 \quad (1 \leq i \leq 3), \quad x_1 = -x_1^2 + 2, \]  
(66)

whereas the second line gives us
\[ x_{i+1} = -x_i^2 + 2 \quad (5 \leq i \leq 7), \quad x_5 = -x_5^2 + 2. \]  
(67)

Moreover, we have
\[ x_1 + x_2 + \cdots + x_7 + x_8 - 1 = 0. \]  
(68)

Obviously these relations correspond to Eqns. (26) and (31). All of the 28 products \( x_ix_j, (1 \leq i < j \leq 8), \) have the form \( (-x_i - x_j) \) which follows immediately from the definitions. But these products mingle the two lines (64) and (65) with each other, for instance, one has \( x_1x_2 = -x_1 - x_8, x_5x_6 = -x_5 - x_1. \) Thus there is no complete analogue to (27)-(30). Moreover, we can jump from the first group (64) to the second one (65) according to
\[ x_{i+4} = -r^{6i} - r^{-6i} = -x_i^6 + 6x_i^4 - 9x_i^2 - 6 \quad (1 \leq i \leq 4) \]  
(69)

and it is possible to jump back according to
\[ x_{i-4} = -r^{3i} - r^{-3i} = x_i^3 - 3x_i \quad (5 \leq i \leq 8). \]  
(70)

Still the two quadruples \((x_1, x_2, x_3, x_4)\) and \((x_5, x_6, x_7, x_8)\) are each accessible to Vandermonde’s basic ideas. One can factorize Eqn. (63) of degree 8 in two equations each of degree 4 with the roots \( x_1, x_2, x_3, x_4 \) and \( x_5, x_6, x_7, x_8 \), resp. Calculating the coefficients of those equations comes down to forming the products \( x_ix_j \) which are known from the definitions. The resulting equations are
\[ (x - x_1)(x - x_2)(x - x_3)(x - x_4) \]
\[ = x^4 + \tau_1 x^3 - (\tau_1 + 2)x^2 - (2\tau_1 + 3)x - 1 = 0 \]  
(71)

and
\[ (x - x_5)(x - x_6)(x - x_7)(x - x_8) \]
\[ = x^4 + \tau_2 x^3 - (\tau_2 + 2)x^2 - (2\tau_2 + 3)x - 1 = 0 \]  
(72)

with the abbreviations
\[ \tau_1 = -x_1 - x_2 - x_3 - x_4 \]
\[ = r + r^2 + r^4 + r^8 + r^{13} + r^{15} + r^{16} \]  
(73)

and
\[ \tau_2 = -x_5 - x_6 - x_7 - x_8 = r^3 + r^6 + r^{12} + r^{10} + r^5 + r^{11} + r^{14}. \] (74)

These quantities coincide with the ones in Eqn. (61) which is checked very easily. One calculates without difficulty
\[ \tau_1 + \tau_2 = -1, \quad \tau_1 \cdot \tau_2 = -4 \] (75)
and this gives us immediately
\[ \tau_1 = -1/2 + 1/2\sqrt{17}, \quad \tau_2 = -1/2 - 1/2\sqrt{17}. \] (76)

Thus the coefficients in the Eqns. (71) and (72) require the quadratic irrationality \( \sqrt{17} \) only. Further each of those equations can be solved by Vandermonde’s methods, and it remains for us to show that this could indeed be done by iterated square roots. Obviously it will suffice to solve one of the Eqns. (71), (72). To this end we fall back upon Vandermonde’s solution (24) of quartic equations and obtain for Eqn. (71)
\[ x = \frac{1}{4} \left[ 1 + \epsilon_1 \sqrt{\alpha} + \epsilon_2 \sqrt{\beta} + \epsilon_3 \sqrt{\gamma} \right] \] (77)
with \( \epsilon_1, \epsilon_2, \epsilon_3 = \pm 1, \epsilon_1 \epsilon_2 \epsilon_3 = +1 \), the three quantities
\[ \alpha = (x_1 + x_2 - x_3 - x_4)^2, \]
\[ \beta = (x_1 + x_3 - x_2 - x_4)^2, \]
\[ \gamma = (x_1 + x_4 - x_2 - x_3)^2 \]
and the condition
\[ \sqrt{\alpha} \cdot \sqrt{\beta} \cdot \sqrt{\gamma} = 7\tau_1 + 12. \] (78)

\( \alpha, \beta, \gamma \) satisfy a cubic equation the coefficients of which are polynomials in \( \tau_1 \) with rational integer coefficients with regard to (71). In our case the relations (61) allow us to apply the Lemma of our subsection 2.3 to the sequence \( (x_1, x_2, x_3, x_4) \). Of course the reference to the Lemma is unnecessary in principle but it tells us what we will have to calculate. In particular, we see that \( \alpha, \gamma \) are permuted with each other under the cyclic permutation \( (x_1 x_2 x_3 x_4) \) whereas the quantities
\[ \beta \quad \text{and} \quad \frac{\alpha - \gamma}{\sqrt{\beta}} = \frac{4(x_1 - x_3)(x_2 - x_4)}{x_1 + x_3 - x_2 - x_4} \]
are left fixed. Hence the latter ones are rational functions of \( \tau_1 \) with rational coefficients. Straightforward calculations show us that
\[ \alpha = A + B\sqrt{-\tau_1 + 8}, \quad \beta = -\tau_1 + 8, \quad \gamma = A - B\sqrt{-\tau_1 + 8} \]
with polynomials \( A, B \) in \( \tau_1 \) with rational coefficients.
Thus the cubic equation with the roots $\alpha, \beta, \gamma$ turns out to be reducible if one admits coefficients of the form $s + t\sqrt{17}$ with rational numbers $s, t$. From Eqn. (78) it follows that $\sqrt{\alpha} \cdot \sqrt{\gamma}$ has the form $C + D\sqrt{-\tau_1 + 8}$ with polynomials $C, D$ in $\tau_1$ with rational coefficients. Hence $\sqrt{\gamma}$ can be written as $(E + F\sqrt{-\tau_1 + 8}) \cdot \sqrt{\alpha}$ where $E, F$ denote some polynomials in $\tau_1$ with rational coefficients.

In summary, in order to obtain $x_1, x_2, x_3, x_4$ we can make do with only the three square roots $\sqrt{17}, \sqrt{-\tau_1 + 8}$ and $\sqrt{\alpha} = \sqrt{A + B\sqrt{-\tau_1 + 8}}$.

Our last question concerning the 17-gon to be answered is: how to link our construction à la Vandermonde to Gauss’s construction? [Gauss 1801, art. 365], [Reich 2003] The key to the answer is the union of the two sequences (64) and (65) in a single sequence induced by the iteration of the map $r \mapsto r^6$. This way we go forth and back between (64) and (65), and the new sequence of 8 terms will be:

$$x_1, x_5, x_2, x_6, x_3, x_7, x_4, x_8.$$ (79)

We can do it this way since 6 mod 17 is a square root of 2 mod 17 and therefore turns out to be a primitive root mod 17, which can be verified easily. The exponents occurring in the first sequence (64) are just the quadratic residues mod 17 whereas the exponents occurring in the second sequence (65) are the quadratic non-residues mod 17. The $x_i$’s are Gauss’s periods of 2 terms $\times (-1)$. Moreover, the sequence (79) is a “cycle” insofar as the successor of every member $x_i$ has the form $\vartheta(x_i) = -x_i^6 + 6x_i^4 - 9x_i^2 - 6$ with the polynomial $\vartheta(X) = -X^6 + 6X^4 - 9X^2 - 6$. The cycle is actually closed since $\vartheta(x_8) = x_1$. We remember that the sums $x_1 + x_2 + x_3 + x_4 = -\tau_1$ and $x_5 + x_6 + x_7 + x_8 = -\tau_2$ are the two Gaussian periods of 8 terms $\times (-1)$. The 4 sums $x_1 + x_3, x_2 + x_4, x_5 + x_7, x_6 + x_8$ do not occur explicitly in our considerations à la Vandermonde, but they coincide with the 4 Gaussian periods of 4 terms $\times (-1)$. Gauss’s systematic use of these periods is a further advantage of his construction over our construction in addition to the conscious use of a primitive root mod 17. The successive subdivision of the periods into periods of fewer terms gives Gauss a clear guideline to obtain the quadratic equations to be solved. Indeed this allows him to avoid equations of degree 4 or 8 which occur in our approach to $x^{17} - 1 = 0$. Especially, the equalities

$$(x_1 + x_3)(x_2 + x_4) = (x_5 + x_7)(x_6 + x_8) = -1$$
give Gauss the equations
\[(x - (x_1 + x_3))(x - (x_2 + x_4)) = x^2 + \tau_1 x - 1 = 0\]
\[(x - (x_5 + x_7))(x - (x_6 + x_8)) = x^2 + \tau_2 x - 1 = 0.\]

As soon as the periods of 4 terms will be known one would be able to calculate the periods of 2 terms in view of the equations
\[(x - x_1)(x - x_3) = x^2 - (x_1 + x_3)x - (x_6 + x_8) = 0,\]
\[(x - x_2)(x - x_4) = x^2 - (x_2 + x_4)x - (x_5 + x_7) = 0.\]

For further details of Gauss’ cyclotomy theory we refer the reader, of course, to the *Disquisitiones Arithmeticae* and its important unfinished and posthumously published continuation *Disquisitionum circa aequationes puras ulterior evolutio*, [Gauss 1801], [Gauss 1863]. The structure of Gauss’ theory is elucidated in Richard Dedekind’s (1831-1916) excellent review of Bachmann’s book [Bachmann 1872] where Dedekind, in particular, emphasized how important the concept of irreducibility is in this theory and the development of algebra following Gauss, [Dedekind 1873], see also [Neumann 2006].

### 3.2. Gauss and Vandermonde

Gauss happened to know at least Vandermonde’s geometrico-topological paper *Remarques sur des problèmes de situation* (1771) and to think highly of it which is testified twice, namely by his letter to the physician and astronomer Wilhelm Olbers (1758-1840) on October 12, 1802, and a note dated from January 22, 1833, in his papers, see [Olbers 1900], p. 103, [Gauss 1863-1933], vol. V, p. 605, vol. X/2, Abhandl. 4, pp. 46-48, 58. As far as we know, nowhere else can we find any trace of Gauss’s reading of Vandermonde. Neither Dunnington’s list of books that Gauss borrowed from the Göttingen University Library during the years 1795-1798 nor Karin Reich’s recent investigations into Gauss’s relations with France give any further direct hints of his preoccupation with Vandermonde. [Dunnington 2004, pp. 398-404], [Reich 1996] Only Dunnington’s list for the date of January 4, 1797, notices that Gauss borrowed Waring’s *Meditationes Algebraicae* from the Göttingen University Library, [Dunnington 2004], p. 400. [24] Waring’s book is not mentioned in Martha Küssner’s monograph on Gauss’s and his “world of books”, [Küssner 1979]. Moreover, in the

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24The Göttingen University Library has copies of the second (1770) and the third (1782) editions of Waring’s book (third edition with the shelf mark 4 Math. II, 9069 < 3 >, HG-FB). As far as I could see Waring, unfortunately, did not indicate where Vandermonde’s paper was published.
extant personal scientific library of Gauss (kept in the Gauss archives of Niedersächsische Staats- und Universitätsbibliothek Göttingen) there are no writings of Vandermonde.

Though a special detail is of some interest to our considerations. Vandermonde’s paper on “problems of situation” mentioned in the introductory sentence of this subsection was printed in one volume together with his algebraic Mémoire. Any user of this volume could hardly overlook Vandermonde’s algebraic treatise, and the old Göttingen University Library does have a copy of that volume (with the shelf mark Phys. Math. III 2550). There is every reason to believe that this very copy was in Gauss’s own hands. Thus one must assume that he became acquainted with Vandermonde’s algebraic treatise not later than in 1802. Paul Stäckel (1862-1919), expert on all kinds of Gaussiana, discoverer of the Notizenjournal and co-editor of the Werke, even held the opinion that Gauss knew Vandermonde’s Mémoire when he was writing his Disquisitiones Arithmeticae, see [Loewy], p. 195, [Gauss 1863-1933], vol. X/2, Abhandl. 4, p. 58.

Above all, we share Stäckel’s view that Gauss in his theory of cyclotomy was then not influenced by Vandermonde. In favour of this view Stäckel refers the reader to Gauss’s letter to Gerling in 1819, discussed in the preceding subsection. In our opinion the peculiar construction of the 17-gon based on Vandermonde’s ideas could very well have been carried out by any mathematician who would have studied Vandermonde’s Mémoire thoroughly and, first of all, who would have asked how to construct the 17-gon. But with Gauss the construction of the 17-gon appears to have been more a fortunate breakthrough on a broad background (“a corollary of an incomplete theory”) than the final completion of a construction which Gauss would have striven for.

Already at several places in the present paper we discussed Lebesgue’s attempts to make plausible direct links of Gauss to Vandermonde. In summary we could not find those attempts convincing since they do not fit the known sources.

A special comment should be given on the situation in 1808. In that year Lagrange published the new edition of his voluminous Traité de la résolution des équations numériques de tous les degrés where inter alia he reproduced at length Vandermonde’s solution of $x^{11} - 1 = 0$ (though supplemented by the use of $\sqrt{-1}$), [Lagrange 1808], Note XIV, §§ 28-36. On the other hand, in that same year Gauss was writing a manuscript on “pure equations” continuing the Disquisitiones Arithmeticae but left unfinished. [Gauss 1863]

We know this detail from a letter of Gauss to Olbers from July 3, 1808, where Gauss acknowledged to have received a copy of Lagrange’s treatise

25 Communicated by Karin Reich to the author, e-mail from June 8, 2004.
and gave some critical comments on this work. At the latest at that time Gauss should have drawn his attention to Vandermonde’s solution of $x^{11} - 1 = 0$. The more it is surprising that Gauss in his manuscript did not mention Vandermonde although there he also treated the equation $x^{11} - 1 = 0$ applying his own theory to it, [Gauss 1863], artt. 13, 17, see also [Bachmann 1872], pp. 96-98.

There remains the question: why did Gauss not quote Vandermonde’s Mémoire in his published writings or in his papers? Our present knowledge of the diary, of his letters and other documents allows us to give one answer only: we will probably never know.

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References


Euler and Number Theory: A Study in Mathematical Invention

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1. Introduction

When discussing the history of mathematics, André Weil once said that “its first use for us is to put or to keep before our eyes ‘illustrious examples’ of first-rate mathematical work” [1, p. 204] to provide useful insights into the process of mathematical research. Here we present one such example: that of Euler, who turned number theory from an amateur’s playground to a vital part of mathematics.

2. Fermat and Number Theory

Euler’s work in number theory began with Fermat’s conjectures. In Euler’s time, these conjectures could be found in three main sources. The last to be written, but the first to be printed, were the letters that made up the Commercium Epistolicum (1658); next were Fermat’s notes on Bachet’s edition of Diophantus’s Arithmetic, which were compiled by Fermat’s son Samuel (1670); finally, Fermat’s own treatises and some of his correspondence which appeared in Varia Opera (1679).
The *Commercium Epistolicum* came about because of a chance meeting between Fermat and Kenelm Digby. Digby was, among other things, adventurer, courtier, agent provocateur, alchemist, and brewer, but his importance to Fermat was his relationship to the scientists and scholars who would later form the Royal Society. During a diplomatic visit to France in 1656, Digby suffered an attack of kidney stones, and retired to southern France for his health. It was around this time that he met Fermat, and the two began to correspond. Digby played the role of an English Mersenne, passing letters between Fermat, Frenicle de Bessy, van Schooten, Wallis, and Brouncker. Although only a small fraction of the *Commercium* dealt with the theory of numbers, it would be the first widespread publication of Fermat’s conjectures.

Fermat chose to model his version of number theory after Diophantus’s *Arithmetic*, which posed problems and provided a specific solution, usually based on clever algebraic manipulations. For example, Book II, Problem 9 asked to express a number that was the sum of two squares as a sum of two other squares. Diophantus took the number $13 = 2^2 + 3^2$, and assumed (in modern notation) $13 = (x+2)^2 + (2x-3)^2$. This gave rise to a homogeneous quadratic equation with two rational solutions. Unfortunately problems like this gave Wallis and others the impression that number theory consisted only of clever calculations and algebraic reductions, and thus held little in the way of real mathematical interest.

Letter XXXVII (April 7, 1658) contains the first appearance of actual theorems (as opposed to problems):

**Conjecture 1.** There is no right triangle with rational side lengths whose area is the square of a rational number.

This statement is equivalent to “Fermat’s Last Theorem” for $n = 4$. The conjecture immediately following is the “Last Theorem” for the $n = 3$ case:

**Conjecture 2.** There is no integral cube that is the sum of two rational cubes.

Fermat provided no proof of either proposition.

In his last letter (June 1658) Fermat presented propositions for which he claimed he had a “most sound proof.” Among these propositions were:

**Conjecture 3.** Every number is a square or the sum of two, three, or four squares.
Conjecture 4. Every prime $p$ of the form $p = 4n + 1$ is the sum of two squares.

Conjecture 5. Every prime $p$ of the form $p = 3n + 1$ is the sum of a square and three times another square.

Conjecture 6. Every prime $p$ of the form $p = 8n + 1$ or $p = 8n + 3$ is the sum of a square and twice another square.

Fermat declined to provide the proofs. He might have hoped other mathematicians would find their own proofs and in the process discover (as he had) the joys of number theory.

Fermat also posed some unsolved problems. To enlist Wallis’s help, he turned to flattery:

We know that Archimedes did not scorn the propositions of Conon, which were certainly true, but unproven, eventually setting forth true and most subtle demonstrations. Why therefore can we not hope for similar help, a French Conon from an English Archimedes? [4, Vol. I, p. 404]

The propositions which he claimed true “in the fashion of Conon” were the following:

Conjecture 7. Numbers of the form $2^{2^n} + 1$ are prime for all $n$.

Today numbers of this type are known as Fermat numbers, with $F_n = 2^{2^n} + 1$.

Conjecture 8. If $p$ is a prime number of the form $p = 8n - 1$, then $2p$ can be written as the sum of three squares.

Conjecture 9. The product of two prime numbers ending in 3 or 7 and of the form $4n + 3$ is the sum of a square and five times another square.

Wallis declined to pursue number theory, and he speaks for all his contemporaries when he writes in Letter VII:

...I looked upon problems of this nature (of which it is easy to contrive a great many in a little time,) to have more in them of labour than either of Use or Difficulty.[8, Vol. II, p. 766]

Wallis later expressed a belief that nothing of significance rested on the truth or falsity of these conjectures [8, Vol. II, p. 782].

Fermat died in 1665. The notes he made in Bachet’s edition of *Diophantus*
were compiled by his son Samuel, and published in 1670. We will only repeat only one conjecture from Fermat’s *Diophantus*; it is, of course:

**Conjecture 10.** The equation $x^n + y^n = z^n$ has no solution in integers for $n > 2$.

What is interesting is that this particular claim appears only in the *Diophantus*, and nowhere else in Fermat’s writing. Indeed, except for Conjecture 2 in the *Commercium*, the only references to the Last Theorem are indirect, as in his challenge to other mathematicians to *find* a cube (or fourth power) that is the sum of two cubes (or two fourth powers). It is probable that Fermat had a proof (or had the outline of a proof) for the $n = 3$ and $n = 4$ cases; it seems unlikely that he had a general proof.

The last major source of Fermat’s number theory appeared in 1679, when the *Varia Opera* appeared. This work includes some of Fermat’s correspondence, as well as most of his mathematical treatises. In it, Fermat repeats Conjecture 7 in several letters, though it is only in the *Dissertation tripartie* that we see why Fermat considered the conjecture important. Fermat argued that the problem of inserting $p - 1$ mean proportionals between two given numbers would be maximally difficult if $p$ was prime; hence he was interested in a formula that would generate arbitrarily large primes. Fermat’s argument is spurious, but it is interesting to note that Fermat numbers do play a role in constructibility in a manner entirely different from that envisioned by Fermat.

If we judge the value of a mathematical discovery by how often it is used in subsequent investigations (the “citation” principle), then Fermat’s most important discovery appeared in the *Varia Opera* as part of a letter dated October 18, 1640 to Frenicle de Bessy:

**Conjecture 11.** Every prime number divides a power, minus one, of any given number, and the aforementioned power is always a divisor of the prime minus one.

In modern terms if $p$ is prime, then $p$ divides $a^m - 1$ for some $m$, and in that case $m$ divides $p - 1$. Fermat omitted an obvious requirement, that $p$ and $a$ have no common factors. This is a more general form of what is usually referred to as Fermat’s Little (or Lesser) Theorem.

Fermat’s attempt to interest his contemporaries in number theory failed, and while his failure cannot be attributed to any single cause, we will point out three key aspects of Fermat’s presentation. First, he presented number theory as a set of problems to be solved, rather than generalizations to be
made, so it appeared as a collection of unrelated results. Moreover, Fermat gave no reason why number theory might be important, so mathematicians were drawn to other fields, such as analysis, where the applications were much more apparent. Finally, where Fermat presented a proposition, he did so without proof, but at the same time claimed possession of a proof, so number theory appeared to be a guessing game between Fermat and other mathematicians. Thus for sixty-five years, number theory languished.

3. Goldbach and Euler

Christian Goldbach would stimulate Euler’s interest in number theory, but only after attempting and failing to interest others in the subject. On December 18, 1723 Goldbach posed a Diophantine problem to Daniel Bernoulli: To find four numbers such that the pairwise product of any two, plus 1, was a square (this is similar to Diophantus’s Book III, Problem 10: to find three numbers so that their pairwise products, added to a given number, was square). Then on February 2, 1724 Goldbach posed a variant: given one number, to find three more so the pairwise products, plus 1, were squares. Like Fermat, Goldbach posed problems rather than suggested general results, though in his letter of September 13, 1724 to Bernoulli, Goldbach mentioned that Jacques Ozanam (1640-1717) proved the difference of two fourth powers could not be a square (again, equivalent to Fermat’s Last Theorem for \( n = 4 \)).\(^1\) Bernoulli had little interest in pursuing number theory, though on June 29, 1728 he wrote to Goldbach with one of his results:

I will finish with a problem which appears to me very curious and which I have solved. Thus: to find two unequal numbers \( x \) and \( y \) so that \( x^y = y^x \). There is one solution among the whole numbers, namely \( x = 2 \) and \( y = 4 \) (because \( 2^4 = 4^2 \)), but one can give an infinite number of broken [real] numbers which solve this problem. There are other [questions] of this type of which I will say nothing [5, vol. II, p. 262].

Thus like Fermat’s correspondents, Bernoulli took only a passing interest in number theory.

Goldbach finally found a willing investigator in Euler, though it took some effort. As a parting note in a letter of December 1, 1729, Goldbach asked:

\(^1\) Ozanam was the author of a number of excellent mathematics texts, including one that helped to found recreational mathematics.
Do you know of Fermat’s observation that all numbers of the form $2^{2^x-1} + 1$, such as 3, 5, 17, etc., are prime, something that he himself was unable to show, and no one after him has shown [5, Vol. I, p. 10]. Euler, like so many of Fermat’s correspondents, thought the result unimportant: “Probably nothing can be discovered from this observation of Fermat” [5, Vol. I, p. 18]. Indeed, since the result had been obtained empirically, Euler doubted its validity (or the validity of any result obtained solely on the basis of scientific induction).

Goldbach was persistent, however, and tried to encourage Euler to work on the problem by suggesting means of approach. Goldbach’s suggestions were of varying quality. On May 22, 1730 he noted that the remainders, when squares of the terms in an arithmetic sequence were divided by a prime number, formed a periodic sequence, an observation that Euler would use later (though not in connection with the Fermat numbers). He offered the additional observation that if $p \neq 2^n$, then $2^p + 1$ had divisors; he gave $2^{84} + 1$, with divisor 17, as an example.

Meanwhile Euler began to study Fermat’s work, and on June 4, 1730 Euler wrote to Goldbach expressing some enthusiasm for number theory. Euler’s attention was caught by the “not inelegant theorem” that every number could be expressed as the sum of four squares. Euler mentions other Fermat conjectures on the resolution of numbers as the sum of polygonal numbers and cubes, “whose proofs would contribute greatly to analysis” [5, vol. I, p. 24]; hence number theory, while worthy of pursuit on its own merits, could also shed useful insight into other areas of mathematics.

Goldbach’s next “contribution” to the investigation of the Fermat numbers was on June 26, 1730 (June 15 O.S.):

It is likely that the least divisor (1 and the number itself not being considered as divisors for this purpose) of any number of the form $a^{2^x} + 1$ is of the form $n^{2^x} + 1$, but this has not yet been completely examined, except in a single case, namely $x = 1$, which is easy to demonstrate [5, vol. I, p. 26]. Hence, Goldbach notes, a proof of Fermat’s conjecture would follow: if it is true that the least divisor $n^{2^x} + 1$ is of the form $a^{2^x} + 1$, then if $n = 2$, then $a$ can only be 1 or 2, but if $a = 1$, then $1^{2^x} + 1 = 2$, which does not divide $2^{2^x} + 1$, and if $a = 2$, then the least divisor is the number itself, which is thus prime.

Euler pointed out almost immediately (June 25—presumably new style) that Goldbach’s claim is untrue: if $a = 34$ (where $34^2 + 1 = 1157$ has least divisor 13), $a = 76$ (where $76^2 + 1 = 5777$ with least divisor 53), and numerous other cases.

Despite Goldbach’s help, Euler made progress and on November 25, 1731 Euler announced a crucial discovery:
Finally consider the formula $2^n - 1$, which cannot be prime unless $n$ is prime, and consider the cases where $2^n - 1$ is not prime, although $n$ is. These exceptions are $n = 11$, $n = 23$, $n = 83$, and all the remaining primes less than 100 make $2^n - 1$ prime. Indeed, $2^{11} - 1$ can be divided by 23, $2^{23} - 1$ by 47, $2^{83} - 1$ by 167. Upon this is based the not inelegant theorem: $2^n - 1$ can always be divided by $n + 1$, whenever $n + 1$ is a prime number. Thus $2^{22} - 1$ can be divided by 23. Often as well $2^{n/2} - 1$, and indeed $2^{n/4} - 1$ etc., can be divided by $n + 1$, and from this the investigation of the case where $2^n - 1$ is prime is not difficult [5, vol. I, p. 59-60].

Euler is announcing a restricted form of Fermat’s Little Theorem, namely that if $p$ is prime, then $2^{p-1} - 1$ is divisible by $p$. As a more general form of this result appeared in Fermat’s Varia Opera, it seems that Euler’s reading of Fermat’s works has to this point been only cursory. As we shall see, Fermat’s Little Theorem is the easiest and most general path to finding factors of the Fermat numbers, and Euler would return to the Little Theorem many times during his number theoretic investigations.

In the meantime, Euler discovered that Fermat’s conjecture was in fact false, and presented his results to the Academy on September 26, 1732. Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus” (E26) was the first of nearly 100 papers on number theory published by Euler; though mathematically insignificant, it hints at things to come.

E26 begins with a discussion of the possible factors of $a^n + 1$. Euler begins by stating two propositions:

(i) If $n = 2^m + 1$, then $a^n + 1$ has a factor of $a + 1$.
(ii) If $n = p(2^m + 1)$, then $a^n + 1$ has a factor of $a^p + 1$.

Euler gave no proof, but these follow easily from straightforward factorization of $a^n + 1$.

Thus in order for $a^n + 1$ to be prime, $n$ must be a power of 2 and, of course, $a$ must be even. These conditions are necessary but not sufficient, and Euler gives several counterexamples:

(i) $a^2 + 1$ has a factor of 5 whenever $a = 5b \pm 3$.
(ii) $30^2 + 1$ has divisor 17 and $50^2 + 1$ has divisor 41.
(iii) $10^4 + 1$ has divisor 73.
(iv) $6^8 + 1$ has divisor 17.
(v) $6^{128} + 1$ has divisor 257.

What insight might we gain from this list of counter-examples and the disproof of Conjecture 7? From the first counter-example, the observant reader will note

\footnote{Euler omits $n = 37$, though in an earlier letter he noted 223 divides $2^{37} - 1$.}
Jeff Suzuki

\[(5b \pm 3)^2 + 1 = 25b^2 \pm 30b + 3^2 + 1\]

Hence it seems that if one wishes \(a^2 + 1\) to be divisible by some prime \(p\), one need only let \(a = (pb \pm c)\) where \(c^2 + 1\) is divisible by \(p\). For small primes \(p\), trial and error would suffice to find \(c\) so \(c^2 + 1\) is divisible by \(p\). Thus since \(4^2 + 1 = 17\), we have \(a^2 + 1\) divisible by 17 whenever \(a = 17k \pm 4\); since \(9^2 + 1 = 82\) is divisible by 41, we have \(a^2 + 1\) divisible by 41 whenever \(a = 41k \pm 9\). Thus the first two counter-examples can be viewed as direct results of naive number theory—uninteresting results of the very type dismissed by Wallis and others.

But what of examples iii through v, and Euler’s factorization of \(F_5 = 2^{2^5} + 1\)? It seems likely that Euler had already suspected the validity of Conjecture 11 and used it to find potential factors.

If a prime \(p\) divides \(a^n + 1\), then \(p\) divides \((a^n + 1)(a^n - 1) = a^{2n} - 1\), and thus by Conjecture 11 \(2n\) is a divisor of \(p - 1\). Thus \(p\) is a prime of the form \(2nk + 1\). Hence the possible factors of \(10^4 + 1\) are primes of the form \(8k + 1\): the first few primes of this form are 17, 41, and 73, and three trial divisions suffice to find a factor. For \(6^8 + 1\) the possible factors are primes of the form \(16k + 1\): 17 is the first prime of this form, and a single trial division suffices to find a factor. Finally \(61^{128} + 1\) might have prime factors of the form \(256k + 1\), and 257 is again the first of these.

For \(2^{2^5} + 1 = 2^{32} + 1\), Euler only needed to examine primes of the form \(64k + 1\). The first few primes of this form are 193, 257, 449, 577, and the actual factor 641. Thus five trial divisions would have sufficed to find the factor; “Hence [the Fermat numbers are] not a solution to the problem of finding a prime that exceeds any given number” [3, Series 1, Vol. II, p. 3]. It is interesting to note that Fermat could have found a factor of \(2^{2^5} + 1\), and the computations were well within his capabilities (and if not, within those of Frenicle de Bessy, a more assiduous calculator). At least one of them should have been capable of disproving Conjecture 7. That they failed to do so is a minor mystery.

Did Euler in fact use Fermat’s Little Theorem to find a factor of \(F_5\)? Euler’s fame as a calculator makes it plausible that trial division was the method used to find the factor 641. However, there are two pieces of evidence that support Euler’s use of Fermat’s Little Theorem. The first is that Euler mentions that he found the factor through “a long method” that opened the way for similar problems to be resolved: this suggests a general method like Conjecture 11 rather than a method like trial division; the “length” in this case would encompass the discovery of Conjecture 11 as well as its application to finding potential factors. More definitively (though perhaps less reliably), we will see that fifteen years later Euler claims that Conjecture 11 was precisely how he found the factor 641.
At the end of E26, Euler notes that he believes (but has not yet proven) that if $a$, $b$ are not divisible by a prime $n + 1$, then $a^n - b^n$ is divisible by $n + 1$. Consequently $2^n - 1$ is divisible by the prime $n + 1$, which is a specific instance of Conjecture 11. Euler concludes E26 with six “theorems” (theorema) he believes valid, but had not yet obtained a proof. The first was:

**Conjecture 12.** If $n$ is prime, then all powers with an exponent of $n - 1$ will leave a remainder of 0 or 1 when divided by $n$.

This is what most books on elementary number theory call Fermat’s Little Theorem: namely that if $p$ is prime and $a$ is relatively prime to $p$, then $a^{p-1} \equiv 1 \mod p$. We shall refer to this particular conjecture as Euler’s form of Fermat’s Little Theorem.

In addition, Euler stated some generalizations of Conjecture 12:

**Conjecture 13.** If $n$ is prime, then any number raised to the power $n^{m-1}(n-1)$, divided by $n^m$, will have remainder 0 or 1.

**Conjecture 14.** If $m, n, p, q, \ldots$ are distinct primes not dividing $a$, and $A$ is the least common multiple of $m - 1$, $n - 1$, $p - 1$, $q - 1$, $\ldots$, then $a^A$ divided by $mnqp\ldots$ leaves a remainder of 0 or 1.

### 4. Fermat’s Little Theorem: First Proof

If Fermat had a proof of Conjecture 11 (or its special case, Conjecture 12), he did not write it down. Leibniz proved the theorem some time before 1683, but the proof only appears in manuscript and was not brought to light until 1894 [2, Vol. I, p. 59]. Thus Euler was the first to publish a proof. He presented “Theorematum quorundam ad numeros primos spectantium demonstratio” (E54) to the St. Petersburg Academy on August 2, 1736.

By now Euler was firmly convinced that number theory was a mathematical discipline worth pursuing. He is rather less enamored of Fermat’s methods, however, and criticizes Fermat’s lack of proof and reliance on (scientific) induction: after all, Fermat’s conjecture on the primality of numbers of the form $2^{2^n} + 1$ seemed well-supported by observation, but it was nonetheless false. This casts doubt on the validity of all conjectures based on observation.

Euler proves his form of Fermat’s Little Theorem by induction; this may
be the first induction proof to appear in post-Newtonian mathematics (induction had already appeared in some of Pascal’s work and in the work of Levi ben Gerson, as well as Leonardo of Pisa’s Liber Quadratorum). Moreover, he not only gives a proof, but carefully elaborates upon the process by which the proof came about; thus E54 is a good example of mathematical epistemology.

First, Euler shows that $2^{p-1} - 1$ is divisible by any prime $p$; this follows from the binomial expansion

$$
(1 + 1)^{p-1} = 1 + \frac{p-1}{1} + \frac{p-1}{1} \frac{p-2}{2} + \frac{p-1}{1} \frac{p-2}{2} \frac{p-3}{3} + \ldots
$$

Since there are $p$ terms in the series, the number of terms is odd; subtracting 1 leaves an even number of terms, which Euler proceeds to group pairwise:

$$
\frac{p-1}{1} + \frac{p-1}{1} \frac{p-2}{2} + \frac{p-1}{1} \frac{p-2}{2} \frac{p-3}{3} + \ldots = \frac{pp-1}{1} + \frac{pp-1}{1} \frac{p-2}{2} \frac{p-3}{3} + \ldots
$$

Since $p$ is an odd number, the last term is $\frac{p}{1} \frac{p-1}{2} \frac{p-2}{3} \frac{p-3}{4} \ldots = p$, and thus every term in the series is divisible by $p$. Thus $2^{p-1} - 1$ is divisible by $p$.

Unfortunately the proof as given is not amenable to generalization, so Euler sought a different proof inspired by a corollary: If $2^{p-1} - 1$ is divisible by some prime $p$, so is $2^p - 2$; conversely, if $p$ divides $2^p - 2$ and $p \neq 2$, then $p$ must divide $2^{p-1} - 1$. This time the binomial expansion gives us:

$$
(1 + 1)^p - 2 = 1 + \frac{p}{1} + \frac{pp-1}{1} \frac{p-1}{2} + \frac{pp-1}{1} \frac{p-3}{2} \frac{p-3}{3} + \ldots + 1 - 2
$$

where all the remaining terms obviously have a factor of $p$. If $p$ is taken to be an odd prime, then since $p$ divides $2^p - 2 = 2(2^{p-1} - 1)$ and $p$ does not divide 2, then it must divide $2^{p-1} - 1$.

Note that this proof emerged from a corollary to the main result. Although listing corollaries is not new, Euler was more diligent than most in providing an exhaustive listing of the consequences of a theorem. As we shall see, some of the corollaries were trivial and we might classify them as examples, but others played important roles in the proofs of later theorems.

After proving this corollary Euler notes that if $p$ divides $2^{p-1} - 1$, then $p$ will also divide $2^{k(p-1)} - 1$, and thus $p$ divides $4^{p-1} - 1$, $8^{p-1} - 1$, $16^{p-1} - 1$, and so on. It seems that Euler is considering the following path to the proof: we need only show that a prime $p$ divides $a^{p-1} - 1$ when $a$ is prime; consequently $p$ will divide $a^{p-1} - 1$ when $a$ is a power of a prime. If we can then show that $p$ divides $a^{p-1} - 1$ when $a$ is a product of primes or powers of primes, we are done.

The task seems daunting, but the very first step along this path to the proof will reveal a shortcut. We must first prove $3^{p-1} - 1$ is divisible by
some prime \( p \) (not equal to 3). Fortunately Euler has given two proofs of the divisibility of \( 2^{p-1} - 1 \); we can use the second to show, first, if \( p \) divides \( 3^p - 3 \) and \( p \neq 3 \), then \( p \) divides \( 3^{p-1} - 1 \). Again we use the binomial expansion:

\[
(1 + 2)^p = 1 + \frac{p}{1} 2 + \frac{p(p-1)}{2} 4 + \frac{p(p-1)(p-2)}{3} 8 + \ldots + 2^p
\]

Since every term is divisible by \( p \) except the first and last, we have \( 3^p - 2^p - 1 \) divisible by \( p \). But \( 3^p - 2^p - 1 = 3^p - 3 - (2^p - 2) \), and \( p \) divides the last two terms, so \( p \) must also divide \( 3^p - 3 \).

We might be tempted to prove the theorem for \( a = 5 \), but note instead that the proof for \( 3 = 2 + 1 \) depended on the validity of the theorem for 2. This gives us an induction step: If \( a^p - a \) can be divided by a prime \( p \), then so can \( (a + 1)^p - (a + 1) \). The proof of the induction step follows by binomial expansion, and the proof of Fermat’s Little Theorem follows immediately. As a postscript to E54, we note that the induction proof of Fermat’s Little Theorem is not well-known, so it is periodically rediscovered by mathematicians great (Laplace and Cauchy) and insignificant ([7]).

5. Fermat’s Little Theorem: Second Proof

In 1740, pro-Slavic elements gained control of the Russian government, and a purge of the pro-German elements which had dominated Russia for a generation was inevitable. For this and other reasons Euler accepted a position at the Berlin Academy of the Sciences, where he would spend the next twenty-six years. However, Euler maintained his membership in the St. Petersburg Academy, and continued to correspond with Goldbach; much of his work in number theory in this period would be communicated to Goldbach first and only later presented to the Berlin Academy.

Euler’s second proof of Fermat’s Little Theorem first appeared in a letter to Goldbach dated March 6, 1742. First Euler proved that any prime \( p \) divided \( (a + b)^p - a^p - b^p \) using the binomial expansion of \( (a + b)^p \). If \( a = b = 1 \), this implied that any prime \( p \) divided \( 2^p - 2 \), and if \( p \neq 2 \), then \( p \) divided \( 2^{p-1} - 1 \). If \( a = 2, b = 1 \), then \( p \) divides \( 3^p - 2^p - 1 \), but since \( p \) divides \( 2^p - 2 \), then \( p \) must also divide \( 3^p - 3 \) and again, if \( p \neq 3 \), \( p \) divides \( 3^{p-1} - 1 \).

Next, Euler shows that if \( p \) divides \( a^p - a \), then \( p \) divides \( (a + 1)^p - a - 1 \); this is the induction step from his first proof of Fermat’s Little Theorem. Thus if \( p \) does not divide \( a \), then \( p \) divides \( a^{p-1} - 1 \) and Fermat’s Little Theorem, as stated by Euler, follows.
Euler then proves an important result, which will eventually lead to a proof of Conjecture 4: If \( p \) is a prime of the form \( 4n - 1 \), it cannot divide the sum of two squares \( a, b \) that are relatively prime to \( p \). This follows because \( p = 4n - 1 \) must divide \( a^{4n-2} - b^{4n-2} \), and thus it cannot divide \( a^{4n-2} + b^{4n-2} \) (since if it did, it could divide their sum and their differences, and thus \( p \) would divide both \( 2a^{4n-2} \) and \( 2b^{4n-2} \), which is impossible if \( p \) is assumed relatively prime to \( a, b \)). Since \( 4n - 2 = 2(2n - 1) \) (Euler calls this an “odd-even” number, a reference to Greek number theory), then \( a^{4n-2} + b^{4n-2} \) has a factor of \( a^2 + b^2 \). Thus \( p \) cannot divide \( a^2 + b^2 \). Conversely, any prime divisor of the sum of two squares must be a prime of the form \( 4n + 1 \).

These results and proofs, substantially unchanged, were presented to the Berlin Academy on March 23, 1747 as “Theoremata circa divisores numerorum” (E134). Euler opens E134 with a defense of number theory as a legitimate area for mathematical research. In support of this viewpoint, Euler points to the existence of seemingly true but as-yet-unproven propositions in number theory: this establishes the superiority of number theory over, say, geometry, since (by Euler’s argument) the more abstruse truths are also those harder to prove. That these truths seem unimportant misses the point: not only is there value in knowing any truth, but the very act of proof may bring to light methods of proof that can be used in other problems, an idea he first stated in his June 4, 1730 letter to Goldbach.

The proof of Fermat’s Little Theorem, and that the prime factors of \( a^2 + b^2 \) must be of the form \( 4n + 1 \), are essentially the same as those in his letter to Goldbach. Euler continues E134 by classifying potential divisors of \( a^4 + b^4 \) (2 or primes of the form \( 8n + 1 \)) and \( a^8 + b^8 \) (2 or primes of the form \( 16n + 1 \)). With these two specific cases dealt with, the generalization to prime divisors of the form \( a^{2m} + b^{2m} \) is transparent: the only possible divisors are 2 or primes of the form \( p = 2^{m+1}n + 1 \). Hence in the case of Fermat’s claim that \( 2^{2^5} + 1 \), one has only to examine primes of the form \( 64k + 1 \); this, he claims, is how he found the factor 641.

E134 concludes with a number of results on power residues. It is the first extensive treatment of the subject, and Euler’s Theorem 11 is the first to provide a general solution to the congruence \( x^m \equiv 1 \pmod{p} \), namely: If \( a = f^2 \pm (2m + 1)\alpha \) where \( p = 2m + 1 \) is prime and \( f, \alpha \) are arbitrary, then \( p \) divides \( a^m - 1 \). Euler then gives six corollaries to this result, then three examples, which he solves (we will leave the solutions as an exercise for the reader):

(i) Find \( a \) so \( a^2 \pm 1 \) is divisible by 5.
(ii) Find \( a \) so \( a^3 \pm 1 \) is divisible by 7.
(iii) Find \( a \) so \( a^5 \pm 1 \) is divisible by 11.

As with his previous works, Euler begins the process of generalization by proving a specific instance: in this case (Theorem 12) if \( a = f^3 \pm (3m+1)\alpha \),
with $p = 3m + 1$ prime, then $a^{m} - 1$ is divisible by $3m + 1$. The proof of
these two cases allows the generalization to be made (Theorem 13): If $a =
\alpha f^{n} \pm (mn + 1)$, where $p = mn + 1$ is prime, then $a^{m} - 1$ is divisible by
$mn + 1$.

6. The Sum of Four Squares

On June 17, 1751 Euler presented “Demonstratio theorematis Fermatiani
omnem numerum sive integrum sive fractum esse summam quatuor pau-
ciorumve quadratorum” (E242) to the Berlin Academy. In it Euler makes
significant progress towards proving Fermat’s Conjecture 3, though he in
fact proves that all rational numbers can be written as the sum of four
rational squares. Lagrange would provide the finishing touches in a 1770
paper, though his proof was subsequently improved by Euler a few years
later.

Of greater importance is that E242 foreshadows the development of group
theory, a theme that will carry Euler through the next phase of his number
theoretic work. In particular, Euler proved a restricted form of Fermat’s
Little Theorem based not, as in Euler’s first two proofs, on the binomial
expansion, but on group theoretic properties.

Euler considers the remainders when the squares of the integers are di-
vided by some prime $p$. If we consider the sequence of squares
$$1, 4, 9, \ldots, p^2, (p + 1)^2, \ldots, 4p^2, (2p + 1)^2, \ldots$$
it is clear (Theorem 2) that the remainders upon division by $p$ form a se-
quence with period $p$. These remainders will necessarily omit some numbers
less than $p$; in particular, if we consider the first $p - 1$ squares
$$1, 4, 9, 16, \ldots, (p - 4)^2, (p - 3)^2, (p - 2)^2, (p - 1)^2$$
it is clear that the terms equidistant from the ends have the same remainder
on division by $p$. Thus we need only consider the remainders when the
squares of the numbers $1, 2, 3, \ldots, \frac{p - 1}{2}$ are divided by $p$. Euler claims that
there are exactly $\frac{p - 1}{2}$ remainders. Euler did not show this, but a proof is
trivial: if two of the squares have the same remainder, then their difference
is divisible by $p$, so either the sum or difference of their roots is divisible
by $p$; however, this is impossible since the sum and difference are both less
than $p$.

After a few more pages Euler proves that if $r$ is any remainder, then all
powers of $r$ are also remainders (Theorem 5). Since there are only finitely
many remainders, then (Corollary 3) an infinite number of powers of $r$
must have equal remainders on division by $p$. Taking two of these powers
with equal remainders, \( r^m \) and \( r^n \), then \( p \) must divide their difference \( r^m - r^n \), and consequently \( p \) must divide \( r^n(r^{m-n} - 1) \); hence (since \( r < p \)), \( p \) must divide \( r^\lambda - 1 \) for some \( \lambda \). This is a restricted form of Fermat’s Little Theorem.

7. Fermat’s Little Theorem: Third Proof

Euler converted the basic idea in this proof of a restricted form of Fermat’s Little Theorem into a proof of the full theorem, presented on February 13, 1755 to the Berlin Academy. There are three noteworthy facts about “Theoremata circa residua ex divisione potestatum relictum” (E262). First, Euler identifies the theorem as one of Fermat’s, which he had not done previously. In addition, Euler proves the theorem in the form it was originally stated by Fermat. Third and most important, the paper takes several crucial steps towards the development of group theory.

The format of E262 is similar to Euler’s other papers on number theory: each theorem is followed by a number of corollaries, and the corollaries are usually the basis of the proof for a later theorem. Euler focuses on the remainders when a power of \( a \) is divided by a prime \( p \). If \( p \) does not divide \( a \), then \( p \) cannot divide any power of \( a \) (Theorem 1); thus, since there are only \( p - 1 \) possible remainders, then the terms of the infinite sequence 1, \( a \), \( a^2 \), \( a^3 \), \ldots, must include some terms with the same remainder on division by \( p \) (Corollary 2). Several results on the behavior of the remainders follow; Euler then shows how these properties can be used to show that the remainder when \( 7^{160} \) is divided by 641 is equal to 640 “or \(-1\).” It is possible (though by no means certain) that this, rather than direct division, was how Euler identified 257 as a factor of \( 6^{128} + 1 \).

With Theorem 3 Euler takes an important step: If \( a \) is relatively prime to \( p \), a prime number, then there exists an infinite number of terms in the geometric sequence 1, \( a \), \( a^2 \), \( a^3 \) \ldots which will have a remainder of 1 when divided by \( p \), and the exponents of these terms will form an arithmetic sequence. The proof is straightforward: Since there must be at least two terms with the same remainder, say \( a^\mu \) and \( a^\nu \) (where we may assume \( \mu > \nu \)), then their difference \( a^\mu - a^\nu = a^\mu(a^{\mu-\nu} - 1) \) is divisible by \( p \); since no power of \( a \) is divisible by \( p \), then \( a^{\mu-\nu} - 1 \) must be. Letting \( \lambda = \mu - \nu \), then every term in the sequence

\[ 1, a^\lambda, a^{2\lambda}, a^{3\lambda}, a^{4\lambda}, a^{5\lambda}, a^{6\lambda} \ldots \]

must also leave a remainder of 1 upon division by \( p \).

Most of the proofs that follow this point in E262 are based on closure properties and counting arguments, and place minimal reliance on symbolic
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manipulation. For example, to show (Theorem 7) that if \( \lambda \) is the least power for which \( a^{\lambda} \equiv 1 \mod p \), then the remainders when the terms of the sequence

\[ a, a^2, a^3, \ldots, a^{\lambda-1} \]

must be all different, Euler assumes that two are the equal, \( a^\mu \) and \( a^\nu \), where we may assume \( \nu < \mu < \lambda \); hence \( a^\mu - a^\nu \equiv a^\nu(a^{\mu-\nu} - 1) \) is divisible by \( p \), and so is \( a^{\mu-\nu} \), which is impossible since \( \lambda > \mu - \nu \) was assumed the least power to leave a remainder of 1 when divided by \( p \). Moreover, since \( a^{n\lambda} \equiv a^\lambda \mod p \), then the sequence of remainders repeats itself with period \( \lambda \) (Theorem 8). Consequently if \( p \) is a prime number and all numbers less than \( p \) appear as remainders, then \( \lambda = p - 1 \) (Theorem 9).

The proof of Theorem 10 foreshadows another important idea in group theory: that of a coset of a subgroup. In this case, Euler proves that if the number of remainders \( \lambda \) is less than \( p - 1 \), then the number of non-remainders is at least as great as the number of remainders. This follows because if we consider the remainders when the terms of the sequence

\[ 1, a, a^2, a^3, \ldots, a^{\lambda-1} \]

are divided by \( p \), the \( \lambda \) remainders are distinct. By assumption there exists at least one non-remainder \( k \); then the terms of the sequence

\[ k, ak, a^2k, a^3k, \ldots, a^{\lambda-1}k \]

are also distinct and non-remainders. Following this are a number of corollaries and theorems of the form: If \( \lambda < \frac{p-1}{n} \), then \( \lambda \leq \frac{p-1}{n+1} \). It follows that \( \lambda \) must be a divisor of \( p - 1 \) (Theorem 13)—Fermat’s Little Theorem in the form stated by Fermat. As a consequence we have Fermat’s Little Theorem in the form stated by Euler (Theorem 14), that if \( p \) is prime and not a divisor of \( a \), then \( a^{p-1} \equiv 1 \mod p \). Euler, incidentally, did not credit Fermat with having conjectured that this might be true for some divisor \( \lambda \) of \( p - 1 \). As with E134, Euler proves Fermat’s Little Theorem partway through the paper and devotes the rest of the paper to the theory of power residues.

8. Fermat’s Little Theorem: Fourth Proof

On June 8, 1758 Euler presented “Theoremata arithmetic nova methodo demonstrata” (E271) which includes his fourth proof of Fermat’s Little Theorem, though Euler actually proves a generalization now called the Euler-Fermat Theorem. Like E262, the method of proof involves little symbolic algebra and many group theoretic ideas. It is instructive to compare the proofs in E262 with those in E271.
E262 began by considering the remainders when a geometric progression $1, a, a^2, \ldots$ where $a$ was divided by $p$. E271 begins by considering the remainders when an *arithmetic* progression $a, a+d, a+2d, \ldots$ was divided by $p$.

Many of the results in E271 are analogous to those in E262: for example, there must be terms with the same remainder. However there is a crucial difference: if $n$ is relatively prime to the difference $d$, then the $n$ terms in the arithmetic sequence from $a$ to $a+(n-1)d$ must, upon division by $n$, yield every number less than $n$ as a remainder (Theorem 1); we have no such guarantee of a complete set of remainders with the geometric sequence.

Euler takes advantage of the one-to-one correspondence between the remainders 0, 1, 2, \ldots, $n-1$ and the terms of the arithmetic sequence $a$, $a+d$, $a+2d$, \ldots, $a+(n-1)d$ in Theorem 2. If some remainder $r$ is relatively prime to $n$, then the corresponding term of the sequence $a+\nu d$ is also relatively prime to $n$; if $r$ and $n$ have a common factor, then so do $a+\nu d$ and $n$. Hence the number of terms in the sequence $a$, $a+d$, $a+2d$, \ldots, $a+(n-1)d$ relatively prime to $n$ is equal to the number of numbers less than $n$ that are relatively prime to $n$.

Since the number of numbers less than $n$ that are relatively prime to $n$ seems to be important, Euler spends the next few pages delving into the properties of what is now called the Euler $\phi$-function (a notation first used by Gauss). We note in passing that, although arithmetic sequences led Euler to the $\phi$-function, Euler then abandoned arithmetic sequences and returned to the geometric progressions of E262.

The proof of the Euler-Fermat theorem begins with Theorem 7, which is a repeat of a result from E262, namely that given any $x$ relatively prime to $N$, there must be some least power $\nu$ where $x^\nu$ leaves a remainder of 1 when divided by $N$. Euler then shows (Theorem 8) that the remainders when the sequence $1, x, x^2, x^3, \ldots$ are divided by $N$ are closed under multiplication and exponentiation; the proof is by straightforward algebraic manipulation. Thus (Theorem 9) the number of distinct remainders when the powers of $x$ are divided by $N$ is either equal to the number of numbers less than $N$ that are relatively prime to it, or is a divisor of this number; this is a proof based on the coset idea. This implies (Theorem 10—the Euler-Fermat Theorem) that $x^\nu$ leaves a remainder of 1 when $\nu$ equals the number of numbers less than $N$ that are relatively prime to $N$, or some divisor of this number. Since (in modern terms) if $N$ is prime, then $\phi(N) = N - 1$, this implies Fermat’s Little Theorem in the form originally stated by Fermat; as in E262, Euler did not indicate that Fermat suggested this might be true for $\nu$ that divided $\phi(N)$. 
9. Results on Quadratic Forms

Let us now turn to quadratic forms (see Fermat’s Conjectures 4, 5, and 6). On September 9, 1741 Euler communicated further “curious properties” he had discovered about numbers of the form $a^2 \pm mb^2$; additional results followed on August 28, 1742, though he was as yet unable to prove any of them. Most of these conjectures would appear (still unproven) in “Theoremata circa divisores numerorum in hac forma $paa \pm qbb$ contentorum” (E164), presented in 1747 to the Berlin Academy. Euler lists some 59 theorems on quadratic forms but only a handful of proofs. Many of Euler’s conjectures coincide with Fermat’s: Euler’s Theorem 2 is Fermat’s Conjecture 4; Theorem 5 is Conjecture 6, and Theorem 8 is an alternate form of Conjecture 5. One is reminded uncomfortably of Wallis’s complaint: the conjectures do not seem particularly profound, and little of consequence seems to hinge on their truth or falsity.

On May 6, 1747 Euler wrote to Goldbach and announced:

I can now prove that, I. All prime numbers of the form $4n+1$ are the sum of two squares, and also II. All non-primes of the form $4n+1$, provided they have no divisors of the form $4n-1$, are also the sum of two squares [5, Vol. I, p. 415].

Euler’s proof is as follows: first, he shows that the product of two numbers that are each the sum of two squares is likewise the sum of two squares; next, if a number that is the sum of two squares is divisible by another number that is the sum of two squares, then their quotient is likewise the sum of two squares. Both proofs are based on symbolic manipulation. Next Euler shows that if a number divides the sum of two squares relatively prime to one another, then the number itself is the sum of two squares. This is proven using Fermat’s method of infinite descent, and is one of the few places where Euler used this particular method.

Finally, Euler is ready to prove the main result, that all primes $p$ of the form $4n+1$ are the sum of two squares. By Fermat’s Little Theorem, $p$ must divide $a^{4n}-b^{4n}$, so $p$ must divide exactly one of $a^{2n}+b^{2n}$ or $a^{2n}-b^{2n}$. Euler claims (but is as yet unable to prove) that there must be a pair of relatively prime numbers $a$, $b$ where $p$ cannot divide $a^{2n}-b^{2n}$; consequently there exists a sum of squares that $p$ divides, and hence $p$ itself must be the sum of two squares.

This proof, again largely unchanged, was presented to the Berlin Academy on March 20, 1749 as “De numeris, qui sunt aggregata duorum quadratorum” (E228). One especially interesting feature about E228 is that Euler begins by listing all numbers less than 200 that are the sum of two squares as well as the numbers less than 200 that are not the sum of two squares;
he then uses observations on this list to help establish some conjectures (which he then proves).

For example, it is trivial to show algebraically that if $N = a^2 + b^2$, then $N$ must either be divisible by 4, or of the form $8n + 1$ or $8n + 2$. We know, of course, not to expect the converse to be true, but we also know that denying the automatic validity of the converse is a learned, not instinctive, response. Euler simply pointed to the list to provide counterexamples to the converse.

It took a little over a year for Euler to complete the proof; again, the first appearance of the final proof was in a letter to Goldbach (April 12, 1749). In order to prove that there must be some $a, b$ for which $p$ does not divide $a^{2n} - b^{2n}$, Euler considers the sequence

$$1, 2^{2n}, 3^{2n}, 4^{2n}, \ldots, (4n)^{2n}$$

Then the first differences

$$2^{2n} - 1, 3^{2n} - 2^{2n}, 4^{2n} - 3^{2n}, \ldots, (4n)^{2n} - (4n - 1)^{2n}$$

cannot all be divisible by $p$, since if they are, then all the differences are so divisible, and in particular the 2nth difference will be. But the 2nth difference will equal $(2n)!$ which cannot be divisible by $p$ (since by assumption $p = 4n + 1$ is prime). Subsequently Euler presented “Demonstratio theoremati Fermatiani omnem numerum primum formae $4n + 1$ esse summam duorum quadratorum” (E241) to the Academy (October 15, 1750)

Euler’s next major results on quadratic forms make up one of his more interesting papers, “Specimen de usu observationum in mathesi pura” (E256), presented to the Berlin Academy on November 22, 1753. Here Euler makes his method of approaching a problem very clear; ironically, Euler is not able to get very far in his investigations!

Literally translated, E256 is “Examples of the use of observation in pure mathematics.” Among other things, Euler is showing that observation (and conjecture) play a crucial role in the development of a mathematical idea. In E256 Euler lists all numbers less than 500 of the form $a^2 + 2b^2$. From this list he makes eight observations; Observation 7 is the same as Fermat’s Conjecture 6. Euler then proves many of these observations, mainly through clever symbolic manipulation, though Euler’s proof of Theorem 9 (no number of the form $2a^2 + b^2$ is divisible by a prime not of that form) uses Fermat’s method of infinite descent. The stumbling block is showing that every prime $p$ must divide some number of the form $2a^2 + b^2$.

At this point Euler notes that the proof applies to $ma^2 + b^2$ only as long as $\frac{m+1}{4}$ does not exceed 4; worse yet, since $3(1)^2 + 1^2 = 4$ has prime divisor $2 \neq 3a^2 + b^2$, the method only works for $m = 1$ (the sum of two squares) and $m = 2$ (the sum of a square and twice another square). Thus by the
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end, Euler notes he is unable to make significant headway on Fermat’s Conjecture 6.

It took six more years before Euler made further progress on quadratic forms. “Supplementum quorundam theorematum arithmeticon, quae in nonnullis demonstrationibus supponuntur” (E272) was presented to the St. Petersburg Academy on October 15, 1759. As in the earlier papers, Euler is able to show, through clever algebraic manipulation, that every prime divisor of a number of the form $a^2 + 3b^2$ is itself of the same form. Thus if $p$ is a prime of the form $6n + 1$, then $p$ must divide $a^6n - b^6n$, and so $p$ divides either $a^{2n} - b^{2n}$ or $a^{4n} + a^{2n}b^{2n} + b^{4n}$. If $p$ divides the latter factor, we are done, since $f^2 + fg + g^2 = (f + 12g)^2 + 3(12g)^2$. Euler then considers the differences

$$2^{2n} - 1, 3^{2n} - 1, 4^{2n} - 1, \ldots, (6n)^{2n} - 1$$

which cannot all be divisible by $p$, since in that case the $2n$th differences would also be divisible by $p$, and the $2n$th differences are $(2n)!$.

E272 is near the beginning of significant slowdown in Euler’s work on number theory. In 26 years Euler published 25 papers on number theory; during the 1750s alone Euler presented 13 papers on the subject. But during the 1760s, Euler would only present four papers on number theory (one of which had been originally presented in the 1750s). It was not until 1770 that Euler resumed his earlier pace in publishing papers on number theory. That year marked the publication of his Algebra, which contained a proof of the impossibility of integer solutions to $x^3 + y^3 = z^3$.

10. Fermat’s Last Theorem

In his correspondence, Fermat claimed or implied the impossibility for $n = 3$ and $n = 4$, but only a sketch for the proof of the $n = 4$ case has been found. The only reference to the famous “last theorem” in its full generality occurs in the Diophantus of 1670, and it is not until February 25, 1748 that Euler mentions Fermat’s Last Theorem:

Fermat says in his Observations on Diophantus that the equation $x^n = y^n + z^n$ is impossible among the rationals, except for the cases $n = 1$ and $n = 2$; that is, a sum of two cubes cannot be a cube, nor can the sum of two biquadrates be a biquadrate, nor in general can the sum of two higher powers equal a like power [5, Vol. I, p. 446].

Euler had already proven the $n = 4$ case. On June 23 and August 16, 1738, he presented “Theorematum quorundam arithmeticon demonstrationes” (E98). The theorem in question is that neither the sum nor difference of two
biquadratics could be a square (i.e., \( x^4 \pm y^4 = z^2 \) has no integral solutions).

This particular theorem has two corollaries of consequences: first, it implies the impossibility of integral solutions to \( x^4 + y^4 = z^4 \). Second, it proves Fermat’s Conjecture 1. Euler noted that a proof of this last had already been given by Frenicle de Bessy, but it depended on properties of right triangles and was so obscure and convoluted that it took considerable effort to understand. Thus Euler sought and presented a clearer and more analytic proof. The proof is largely accomplished through symbolic algebra and a few parity arguments. There are very few new methods in this proof.

Euler claimed a proof of the \( n = 3 \) case as early as August 4, 1753, but this proof did not appear until his *Algebra* (1770), and the proof presented is incomplete. Portions of the proof in the *Algebra* are reminiscent of work done by Euler in E272; indeed, Euler noted a possible connection between his work in E272 and a proof of Fermat’s Last Theorem. The connection for the \( n = 3 \) case is this: if \( x^3 \pm y^3 = z^3 \), then \( z^3 = (x \pm y)(x^2 \pm xy + y^2) \). But this second factor is \( x^2 \pm xy + y^2 = x^2 \pm xy + \frac{7}{4}y^2 + \frac{3}{4}y^2 = (x \pm \frac{y}{2}) + 3\left(\frac{y}{2}\right)^2 \).

Hence the sum or difference of two cubes factors into a product consisting of the sum or difference of their roots, and a number expressible as \( p^2 + 3q^2 \). We may assume that \( x, y, \) and \( z \) have no common factors; hence one factor of \( z \) must be a number of the form \( p^2 + 3q^2 \). Euler assumed without proof that numbers of this form are products of primes of this form. In a like manner, other quadratic forms may play a role as factors of numbers of the form \( x^n \pm y^n \). Unfortunately neither Euler nor anyone else would be able to convert this speculation into a viable proof.

For the \( n = 3 \) case Euler factored \( p^2 + 3q^2 \) as \((p + q\sqrt{-3})(p - q\sqrt{-3})\). Through the use of this variant of Gaussian integers, and an implicit assumption of unique factorization, Euler was able to construct a descent proof that showed the impossibility of the \( n = 3 \) case. As in his proof for the \( n = 4 \) case, Euler’s proof rested primarily on symbolic manipulations and parity arguments, and introduced little in the way of new methods.

The *Algebra* marked Euler’s return to number theory, and soon he would return to and even exceed the pace he set before his slow decade of the 1760s. But while the pre-1760 papers broke new ground, established powerful new tools, and reinvented the subject as one fit for serious mathematical investigation, the post-1770 papers were, by and large, of little significance. Fortunately Euler’s work had inspired a new convert to number theory: his successor at Berlin, Joseph Louis Lagrange. Lagrange and later Legendre would carry the investigation of number theory forward until the age of Gauss.
References

In addition to all his other mathematical achievements, Euler discovered the first significant results in many fields of modern combinatorics. In this chapter, we survey this work spread over some fourteen publications.

In the first section, we consider Euler’s work on partitions of integers, focusing on three articles and a book that span his career. The second section addresses various types of squares that Euler considered — magic, Graeco-Latin, and chessboards. The final section samples Euler’s contributions to the study of binomial coefficients, the Catalan numbers, derangements, and the Josephus problem. We omit the bridges of Königsberg and the polyhedral formula as they are treated elsewhere in this volume.

1. Partitions

In 1699 Leibniz wrote to Johann Bernoulli asking about “divulsions of integers,” now called partitions. A basic problem is determining the number $p(n)$ of ways that a positive integer $n$ can be written as the sum of positive integers; for example, $p(4) = 5$, corresponding to the sums $4$, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. Variations of this basic problem ask for partitions
of $n$ into a given number of parts, or into distinct parts, odd parts, etc. For example, we can write the number 10 as the sum of exactly three positive numbers in eight ways,

$$
8 + 1 + 1 \quad 7 + 2 + 1 \quad 6 + 3 + 1 \quad 6 + 2 + 2 \\
5 + 4 + 1 \quad 5 + 3 + 2 \quad 4 + 4 + 2 \quad 4 + 3 + 3
$$

Notice that four of these are partitions with distinct parts.

The first publication on partitions of integers came from a presentation that Euler made in 1741 to the St. Petersburg Academy [E158]. Euler answered two questions posed by Philip Naudé and stated what became known as the pentagonal number theorem. We present his arguments from a later publication, his celebrated *Introductio in Analysin Infinitorum* [E101], in which his main method of proof is generating functions; Euler often repeated his results on partitions, in some cases providing multiple proofs.

Naudé’s Question 1: In how many ways can the number 50 be written as the sum of seven distinct positive integers? To answer this, Euler considered the following infinite product in $x$ and $z$, organized in increasing powers of $z$.

$$(1 + x z) (1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z)(1 + x^6 z) \cdots$$

$$= 1 + z(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots)$$

$$+ z^2(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \cdots) \quad (1)$$

$$+ z^3(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + \cdots)$$

$$+ \cdots$$

What does a term such as $4x^{10}z^3$ indicate? Each $x^{10}z^3$-term arises from one of the four products $x^7z \cdot x^2z \cdot xz$, $x^6z \cdot x^3z \cdot xz$, $x^5z \cdot x^4z \cdot xz$, and $x^5z \cdot x^3z \cdot x^2z$. These products correspond to the above four ways that we can write 10 as a partition of three distinct positive integers.

However, we do not want to have to compute the coefficient of $x^{50}z^7$ from the terms of this infinite product. Writing $m^{(i)}$ for the number of ways of writing $m$ as the sum of $\mu$ “inequal” integers, Euler established the following recurrence relation:

$$(m + \mu)^{(i)} = m^{(i)} + m^{(i-1)}$$

With this, it is not hard to compute 522 as the answer to Naudé’s first question.

Naudé’s Question 2: In how many ways can the number 50 be written as the sum of seven positive integers, equal or unequal? Here Euler con-
sidered another infinite product in $x$ and $z$, this time with factors in the
denominator.

\[
\frac{1}{(1-xz)(1-x^2z)(1-x^3z)\cdots} = \left(\frac{1}{1-xz}\right)\left(\frac{1}{1-x^2z}\right)\left(\frac{1}{1-x^3z}\right)\cdots
\]

\[
= (1 + xz + x^2z^2 + x^3z^3 + \cdots)(1 + x^2z + x^4z^2 + \cdots)\cdots
\]

\[
= 1 + z(x + x^2 + x^3 + x^4 + x^5 + x^7 + x^8 + \cdots)
\]

\[
+ z^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + \cdots)
\]

\[
+ z^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + \cdots)
\]

\[
+ \cdots
\]

Here, a term such as $8x^{10}z^3$ indicates the eight ways (given above) that
we can write 10 as the sum of three positive integers. Again, there is a
recurrence relation. Writing $m^{(\mu)}$ when the parts need not be distinct,
Euler established the following equation:

\[
m^{(\mu)} = (m - \mu)^{(\mu)} + (m - 1)^{\mu-1}
\]

From this, we can determine that the answer to Naudé’s second question
is 8496. But Euler first established a connection between the two questions.
He deduced the number of partitions with $\mu$ distinct parts from the formula

\[
m^{(\mu)} = \left(\frac{m - \mu(\mu - 1)}{2}\right)^{(\mu)}
\]

— in particular, the number of unrestricted seven-part partitions of 50 is
equal to the number of distinct seven-part partitions of $50 + 21 = 71$.
Euler also discussed the connection between the numbers of parts in a and
the maximum number of parts; for example, 8496 is also the number of
partitions of $50 - 7 = 43$ that use only the numbers 1, 2, \ldots, 7.

This revolutionary paper ends with a celebrated formula that Euler had
mentioned in his correspondence (see [B]), but had not yet proved. If we
let $z = 1$ in equation (2) we can combine the expressions in $x$ to give

\[
\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}
\]

\[
= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^6 + 11x^7 + 15x^8 + 22x^9 + \cdots
\]

where the coefficient of $x^k$ is the total number of unrestricted partitions
of $k$. But now consider the reciprocal of this infinite product. From extensive
computations Euler concluded that
where the exponents are the generalized pentagonal numbers, \((3k^2 \pm k)/2\). This result is now known as Euler’s pentagonal number theorem.

Euler devoted a chapter of his 1748 Introductio [E101] to partitions, expanding on the results from the previous article. It includes one of the most striking and elegant applications of generating functions to partitions. Letting \(z = 1\) in equation (1) and combining the expressions in \(x\) we obtain

\[ Q = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \cdots \]

\[ = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \cdots \]

where the coefficient of \(x^k\) is the total number of partitions of \(k\) into distinct parts. Again, we consider the reciprocal of this infinite product. With the infinite product \(P\) as defined in equation (3), we note that the terms of

\[ PQ = (1 - x^2)(1 - x^4)(1 - x^6) \cdots \]

are factors of \(P\), so that we can divide \(P\) by \(PQ\). This leaves

\[ \frac{P}{PQ} = \frac{(1 - x)(1 - x^2)(1 - x^3)\cdots}{(1 - x^2)(1 - x^4)(1 - x^6)\cdots} = (1 - x)(1 - x^3)(1 - x^5)\cdots = \frac{1}{Q} \]

so that \(Q\) can now be written as

\[ Q = \frac{1}{(1 - x)(1 - x^3)(1 - x^5)\cdots} \]

\[ = \left( \frac{1}{1 - x} \right) \left( \frac{1}{1 - x^3} \right) \left( \frac{1}{1 - x^5} \right) \cdots \]

\[ = (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)\cdots \]

\[ = (1 + x + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots)\cdots \]

in which the coefficient of \(x^k\) gives the number of partitions of \(k\) into odd parts, not necessarily distinct. This proves a surprising theorem:

The number of partitions of \(k\) into distinct parts equals the number of partitions of \(k\) into odd parts.

As an example, note that there are six partitions of the number 8 into distinct parts \((8, 7 + 1, 6 + 2, 5 + 3, 5 + 2 + 1, \text{ and } 4 + 3 + 1)\) and six partitions of 8 into odd parts \((7 + 1, 5 + 3, 5 + 1 + 1 + 1, 3 + 3 + 1 + 1, 3 + 1 + 1 + 1 + 1, \text{ and } 1 + 1 + 1 + 1 + 1 + 1 + 1)\).

The chapter concludes with generating-function proofs of the facts that each positive integer can be expressed uniquely as a sum of distinct powers
of 2 and can also be uniquely expressed as a sum and difference of distinct powers of 3.

Euler continued his exploration of partitions with a paper presented in early 1750 [E191]. This is his longest article on partitions, filled not so much with new material, but rather with numerous examples and tables. Writing \( n^{(\infty)} \) for what is now known as \( p(n) \), he established that

\[
n^{(\infty)} = (n - 1)^{(1)} + (n - 2)^{(2)} + (n - 3)^{(3)} + \cdots + (n - n)^{(n)}
\]

which he then used recursively to suggest a recurrence relation for \( n^{(\infty)} \); this also follows from the still unproved pentagonal number theorem (3).

Andrews [A] asserts that “No one has ever found a more efficient algorithm for computing \( p(N) \). It computes a full table of values of \( p(n) \) for \( n \leq N \) in time \( O(N^{3/2}) \).”

Later in 1750, in a letter to Christian Goldbach, Euler finally proved the pentagonal number theorem (3). He eventually published two proofs, and also considered properties of the pentagonal numbers themselves, such as that each pentagonal number is one-third of a triangular number; see Bell [B] for a detailed account of this pentagonal-number result throughout Euler’s work. Interestingly, the function that sums the divisors of a number — for example, \( \sum 10 = 1 + 2 + 5 + 10 = 18 \) in Euler’s notation — shares almost the same recurrence relation; Euler also devoted several articles to this divisor function.

Euler returned to partitions once more in a presentation of 1768 [E394], in which he combined two previous restrictions on partitions — the number of parts and how large each part can be. The running example for much of the article used only 1, 2, \ldots, 6 as parts, and Euler’s computation of the coefficients in \((x + x^2 + \cdots + x^6)^n\) was simplified by the use of various recurrence relations; for example, Euler established that

\[
n^{(6)} = \frac{(n - 1)(n - 1)^{(6)} - (48 - n)(n - 6)^{(6)} - (43 - n)(n - 7)^{(6)}}{n - 6}
\]

as an example of the formulas that can be derived from these methods.

The article also considers the problem of partitions with varying constraints. In particular, Euler considered three-part partitions where the first part is 6 or less, the second is 8 or less, and the third is 12 or less. The 576 resulting partitions of the numbers from 3 to 26 are specified by the coefficients of

\[
(1 + x + \cdots + x^6)(1 + x + \cdots + x^8)(1 + x + \cdots + x^{12})
\]
whose computations are simplified by generating-function techniques. The coefficients for $x^k$, $k = 3, 4, \ldots, 14$ are as follows:

<table>
<thead>
<tr>
<th>exponent</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>coefficient</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>27</td>
<td>33</td>
<td>38</td>
<td>42</td>
<td>45</td>
<td>47</td>
</tr>
</tbody>
</table>

The coefficients of $x^k$ for $k = 15, 16, \ldots, 26$ are the reverse of these, from 47 down to 1.

Although Euler was not the first mathematician to consider generating functions or partitions of integers — De Moivre [dM] had used generating functions in 1718 to analyze multiple-step recurrence relations — he was the first to treat them in a thorough and general way. A thorough early history of partitions, drawing on some of Euler’s correspondence and including work of his contemporaries, is given in Dickson [D]. Generating functions have since become an essential tool in combinatorics and number theory, “a clothes line on which we hang up a sequence of numbers for display” (see Wilf [W]). Even though Hardy and Ramanujan obtained a stunning exact formula for the partition number $p(n)$, the theory of partitions has remained a very active area of research with many impressive results and many outstanding problems (see Andrews and Eriksson [AE]). Even so, we can still agree with Andrews [A] that “Almost every discovery in partitions owes something to Euler’s beginnings.”

### 2. Squares

In 1776, Euler delivered a short article *On magic squares* to the St. Petersburg Academy [E795]. Such an arrangement of integers, already long familiar, is an $n \times n$ square with the numbers $1, 2, \ldots, n^2$ arranged in such a way that the numbers in each row, each column, and each of the two diagonals have the same sum. After discussing what this sum must be, Euler introduced Latin and Greek letters to help him to analyze magic squares: each Latin letter stands for a multiple of $n$ from 0 to $n(n-1)$ and each Greek letter has a value from 1 to $n$. With each individual cell assigned both a Latin and a Greek letter in such a way that no pair is repeated, he was able to determine values for the letters so that the sums give a magic square. An example of such a $3 \times 3$ square is given in Table 1. Note that the left-hand square has each letter occurring exactly once in each row and column, and is an example of what is now known as a Graeco-Latin square (because of Euler’s notation).

A $4 \times 4$ non-Graeco-Latin square appears in Table 2, with the associated magic square obtained by using the specified values. Although the square is
not Graeco-Latin (notice that $\alpha$ appears twice in the first column, $a$ twice in the first row), it still yields a magic square. Euler's article closes with descriptions (depending on whether $n$ is even or odd) of how to construct magic squares of any size.

Euler remained interested in the problem of Graeco-Latin squares. Three years later, in 1779, he presented one of his longest published papers, *Investigations on a new type of magic square* [E530]. It begins with the celebrated "36-officers problem:"

*Six regiments are each represented with six officers, one per rank — can they be placed in a 6 by 6 formation such that there is one officer of each regiment in each row and column, and one officer of each rank in each row and column?*

Euler claimed that the answer is no, and embarked on a thorough study of Graeco-Latin and Latin squares.

By the second page, Euler had replaced his Graeco-Latin notation with pairs of numbers, the second written in superscript; we give an example in Table 3. He gave several general methods for building Latin squares, of which the "double march" is illustrated in Table 3 on the right — notice how the square divides into four smaller Latin squares involving 1 and 2 or 3 and 4. There are also single, triple, and quadruple marches.

He also used these methods (and others) to build Graeco-Latin squares of odd order and orders that are multiples of 4; however, none of the methods produced a $6 \times 6$ Graeco-Latin square. Euler suspected that there are none, claiming that he would have come across one in his investigation if any existed, while recognizing that an exhaustive search would be very lengthy. There is no formal conjecture of the general case, but he stated

---

**Table 1**

Graeco-Latin square for $n = 3$ and associated magic square from $a = 0, b = 6, c = 3; \alpha = 1, \beta = 3, \gamma = 2$.

| $a\gamma$ | $b\beta$ | $c\alpha$ | 2 | 9 | 4 |
| $b\alpha$ | $c\gamma$ | $a\beta$ | 7 | 5 | 3 |
| $c\beta$ | $a\alpha$ | $b\gamma$ | 6 | 1 | 8 |

---

**Table 2**

Non-Graeco-Latin square for $n = 4$ and associated magic square from $a = 0, b = 4, c = 8, d = 12; \alpha = 1, \beta = 2, \gamma = 3, \delta = 4$.

| $a\alpha$ | $a\delta$ | $d\beta$ | $d\gamma$ | 1 | 4 | 14 | 15 |
| $d\alpha$ | $d\delta$ | $a\beta$ | $a\gamma$ | 13 | 16 | 2 | 3 |
| $b\delta$ | $b\alpha$ | $c\gamma$ | $c\beta$ | 8 | 5 | 11 | 10 |
| $c\delta$ | $c\alpha$ | $b\gamma$ | $b\beta$ | 12 | 9 | 7 | 6 |
that a Graeco-Latin square of order $4k + 2$ would have to be “completely irregular” and seemed to doubt that there are such. Euler’s article also includes enumeration of Latin squares of small orders under certain conditions and a discussion of collections of Latin squares any two of which can be combined into a Graeco-Latin square. For example, notice in Table 3 that the Latin square of base numbers on the left can be combined with the Latin square on the right to make a Graeco-Latin square, and the same is true for the Latin square of superscript numbers.

An exhaustive search in the next century verified Euler’s conjecture for $6 \times 6$ Graeco-Latin squares, showing that Euler was right about the 36 officers problem. However, the general $4k + 2$ conjecture was shown to be false in 1960 by Bose, Shrikhande and Parker (see Klyve and Stemkoski [KS] for details); this result was so unexpected that it was reported on the front page of the New York Times. A related research area is that of finding “mutually orthogonal Latin squares:” are there $n - 1$ Latin squares of size $n \times n$ with the property that any two of them constitute a Graeco-Latin square? (The three $4 \times 4$ Latin squares of Table 3 are an example.) This is an area of contemporary research (see Mullen [M]).

Recently, there have been uninformed claims in the media that Euler invented the popular number puzzle Sudoku in which $9 \times 9$ Latin squares satisfy the additional requirement that no number should be repeated in the principal $3 \times 3$ subsquares. While a completed Sudoku puzzle is a Latin square, none of Euler’s $9 \times 9$ examples of Latin squares has the form of a Sudoku puzzle. The closest he came were the $4 \times 4$ examples of Table 3 (each set of numbers in the left-hand square), which coincidentally have the additional structure that each $2 \times 2$ corner contains 1, 2, 3, and 4.

However, Euler did write on a topic in recreational mathematics that relates to squares. His paper Solution of a curious question that does not seem to have been subject to any analysis is based on a 1759 presentation to the Berlin Academy about knight’s tours on chess boards of various sizes [E309]. The question is how to have a knight make its L-shaped moves around the board and visit each square exactly once (now known as a Hamiltonian cycle!). Euler demonstrated many such tours on standard $8 \times 8$ and other size boards, often producing tours with high degrees of

<table>
<thead>
<tr>
<th>$1^1$</th>
<th>$4^3$</th>
<th>$2^4$</th>
<th>$3^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^2$</td>
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<tr>
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<td>$4^4$</td>
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<td>$2^3$</td>
</tr>
<tr>
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<tr>
<td>$2$</td>
<td>$1$</td>
<td>$4$</td>
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<tr>
<td>$3$</td>
<td>$4$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td>$4$</td>
<td>$3$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Table 3
Revised notation for Graeco-Latin squares, and example of a “double march.”
symmetry. A knight’s tour is shown by labeling consecutive positions, as in Tables 4 and 5.

<table>
<thead>
<tr>
<th>37</th>
<th>62</th>
<th>43</th>
<th>56</th>
<th>35</th>
<th>60</th>
<th>41</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>55</td>
<td>36</td>
<td>61</td>
<td>42</td>
<td>49</td>
<td>34</td>
<td>59</td>
</tr>
<tr>
<td>63</td>
<td>38</td>
<td>53</td>
<td>46</td>
<td>57</td>
<td>40</td>
<td>51</td>
<td>48</td>
</tr>
<tr>
<td>54</td>
<td>45</td>
<td>64</td>
<td>39</td>
<td>52</td>
<td>47</td>
<td>58</td>
<td>33</td>
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<tr>
<td>1</td>
<td>26</td>
<td>15</td>
<td>20</td>
<td>7</td>
<td>32</td>
<td>13</td>
<td>22</td>
</tr>
<tr>
<td>16</td>
<td>19</td>
<td>8</td>
<td>25</td>
<td>14</td>
<td>21</td>
<td>6</td>
<td>31</td>
</tr>
<tr>
<td>27</td>
<td>2</td>
<td>17</td>
<td>10</td>
<td>29</td>
<td>4</td>
<td>23</td>
<td>12</td>
</tr>
<tr>
<td>18</td>
<td>9</td>
<td>28</td>
<td>3</td>
<td>24</td>
<td>11</td>
<td>30</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 4
Closed knight’s tour of an 8 × 8 board with half-turn symmetry about the center of the board.

For an \( n \times n \) board, the labels are \( 1, 2, \ldots, n^2 \) — could a knight’s tour give rise to a magic square? This is not a question Euler posed; the closest such path given in the article is the \( 5 \times 5 \) example of Table 5: the diagonals, as well as rows and columns including the center, all sum to 65, but this seems coincidental. Computers have recently been used to conclude that there are no \( 8 \times 8 \) “Euler knight tours,” but if the requirement about diagonal sums is removed, then there are 140 such tours (see Jelliss [J]).

<table>
<thead>
<tr>
<th>7</th>
<th>12</th>
<th>17</th>
<th>22</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>23</td>
<td>6</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>25</td>
<td>4</td>
<td>21</td>
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<tr>
<td>24</td>
<td>19</td>
<td>2</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>9</td>
<td>20</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5
Non-closed knight’s tour of a \( 5 \times 5 \) board.

3. Other Topics

**Binomial coefficients**

Many of Euler’s articles incorporate binomial coefficients; here we highlight three articles that consider properties of these numbers with integer arguments.
In a 1776 presentation to the St. Petersburg Academy [E575], primarily about integrals, Euler collected several facts about binomial coefficients, using notations very similar to those used today. One primary result, in modern notation, is the following equation:

\[
\binom{n}{0} \left( \frac{p}{q} \right) + \binom{n}{1} \left( \frac{p}{q+1} \right) + \cdots = \binom{p+n}{q+n}
\]

Later in the same year, Euler presented an article generalizing binomial coefficients to higher-degree polynomials [E584]. He first reviewed the relationship between

\[
\binom{n}{p}
\]

and the coefficients of \((1 + z)^n\), and properties such as the sum of squares (a special case of the preceding formula) and

\[
\binom{n+1}{p+1} = \binom{n}{p} + \binom{n}{p+1}
\]

Euler then moved on to trinomial, quadrinomial, and higher-order coefficients. In particular, the coefficients of \((1+z+z^2+z^3)^n\) (to use his notation for squares) for small values of \(n\) are given in the following partial table, where the columns correspond to the degree of \(z\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>2</td>
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<tr>
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<td>1</td>
<td>4</td>
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<td>40</td>
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<tr>
<td>5</td>
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<td>15</td>
<td>35</td>
<td>65</td>
<td>101</td>
<td>135</td>
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<td>21</td>
<td>56</td>
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<td>216</td>
<td>336</td>
<td>456</td>
<td>546</td>
<td>580</td>
<td>546</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 6
Coefficient of \(z^k\) in \((1+z+z^2+z^3)^n\), the quadrinomial coefficients.

He showed that these coefficients, indexed here with 4, satisfy

\[
\binom{n+1}{p+3} = \binom{n}{p+3} + \binom{n}{p+2} + \binom{n}{p+1} + \binom{n}{p}
\]

and the general relationship

\[
\binom{n}{0} \binom{m}{0} + \binom{n}{1} \binom{m}{1} + \cdots = \binom{n+m}{3n}
\].
In 1778, Euler returned to these coefficients in another presentation in the same venue [E709]. By writing

\[(1 + z + zz + z^3)^n = (1 + z(1 + z + zz))^n,\]

he related the quardrinomial coefficients to the binomial and trinomial ones. For example, again writing subscripts for the degree so that binomial coefficients are indexed by 2, we have

\[
\binom{n}{4} = \binom{n}{2} \binom{4}{0} + \binom{n}{3} \binom{3}{1} + \binom{n}{2} \binom{2}{2} + \binom{n}{1} \binom{1}{3}
\]

\[
= \binom{n}{4} + 3 \binom{n}{3} + 3 \binom{n}{2}
\]

Catalan numbers

In a letter of 4 September 1751 to Christian Goldbach [EG], Euler discussed the problem of finding the number of different ways that a polygon can be broken into triangles using diagonals. After considering several examples, he gave the formula

\[
\frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n - 10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n - 1)}
\]

which we now call the \((n-2)\)nd Catalan number, usually written as

\[
\frac{1}{n-1} \binom{2n-4}{n-2}.
\]

Euler closed his letter with the generating function associated with this sequence, a topic that he and Goldbach discussed in subsequent correspondence:

\[
1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \cdots = \frac{1 - 2a - \sqrt{1 - 4a}}{2a^2}
\]

Derangements

Many of Euler’s articles discuss probability and games of chance, especially lotteries. One that is relevant here is his *Calculation of the probability in the game of coincidence* [E201], published in 1753. Two players have identical decks of cards, shuffled, which they turn over one at a time. If they turn over the same card at any turn, the first player wins and the game ends. The second player wins only if the cards are different at every
turn. Euler explained that this is equivalent to numbering the cards and checking to see if the second player turns over card $n$ on turn $n$, and showed that the probability of the second player winning is

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \cdots = \frac{1}{e}$$

In 1779, Euler returned to this topic with *The solution of a curious question in the science of combinations*, presented to the St. Petersburg Academy [E738]. No longer motivated by the card game, he asked for the number of ways that the sequence $a, b, c, d, e, \ldots$, can be reordered such that no letter is in its original position. We will write $D(n)$ for this, suggesting the later name “derangement” for such an ordering. Euler derived the following two identities, and showed their equivalence:

$$D(n) = (n - 1)(D(n - 1) + D(n - 2)),$$
$$D(n) = nD(n - 1) + (-1)^n$$

*The Josephus problem*

We close with another topic in recreational mathematics, a staple of discrete mathematics textbooks. Suppose that $n$ people stand in a circle. Moving clockwise, we remove every $k$th person. Which person is the last to be removed? This is known as the Josephus Flavius problem, named for the Jewish historian and general and an intricate suicide pact which left him the last man standing. In *Observations about a new and singular type of progression*, presented to the St. Petersburg Academy in 1771 [E476], Euler included several tables of data. For instance, with fifteen people, removing every fourth one gives the following order of removal:

$$4, 8, 12, 1, 6, 11, 2, 9, 15, 10, 5, 3, 7, 14, 13$$

He then analyzed the general problem to develop a recursive procedure for determining the number of the last person removed. In many cases the recursive step is just adding the number skipped to the previous answer. To demonstrate that the procedure is feasible, Euler gave the computations to show that if there are 5000 people and every ninth person is removed, then the last one standing is number 4897.

**Note:** Euler’s publications are cited below by their Eneström number. All are reprinted in *Leonhard Euleri Opera omnia*, abbreviated *OO*. Most are available electronically at *The Euler Archive*, http://eulerarchive.org, which also links to some English translations.
References


B. J. Bell, Euler and the pentagonal number theorem, Mathematics ArXiv HO/0510054, 2005.


E709. —, *De evolutione potestatis polynomialis cuiuscunque \((1 + x + x^2 + x^3 + x^4 + \text{ etc.})^n\)*, *Nova acta Petrop.* 12 (1801) 47–57. Reprinted in *OO*, I.16, 28–40.


J. G. Jelliss Knight’s Tour Notes, http://home.freenet.net/ktn/


The Truth about Kônigsberg

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USA  
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Milton Keynes  
UK

Euler’s 1736 paper on the bridges of Kônigsberg is widely regarded as the earliest contribution to graph theory—yet Euler’s solution made no mention of graphs. In this paper\textsuperscript{1} we place Euler’s views on the Königsberg problem in their historical context, present his method of solution, and trace the development of the present-day solution.

1. What Euler didn’t do

A well-known recreational puzzle concerns the bridges of Königsberg. It is claimed that in the early eighteenth century the citizens of Königsberg used to spend their Sunday afternoons walking around their beautiful city. The city itself consisted of four land areas separated by branches of the river Pregel over which there were seven bridges, as illustrated in Figure 1.

\textsuperscript{1} This article has previously appeared in The College Mathematics Journal, 35(3), pp. 198–207. It won The Mathematical Association of America’s George Pólya Award in 2005. It is included here with the kind permission of The Mathematical Association of America.
The problem that the citizens set themselves was to walk around the city, crossing each of the seven bridges exactly once and, if possible, returning to their starting point.

If you look in some books on recreational mathematics, or listen to some graph-theorists who should know better, you will 'learn' that Leonhard Euler investigated the Königsberg bridges problem by drawing a graph of the city, as in Figure 2, with a vertex representing each of the four land areas and an edge representing each of the seven bridges. The problem is then to find a trail in this graph that passes along each edge just once.

But Euler didn’t draw the graph in Figure 2—graphs of this kind didn’t make their first appearance until the second half of the nineteenth century. So what exactly did Euler do?
2. The Königsberg bridges problem

In 1254 the Teutonic knights founded the Prussian city of Königsberg (literally, king’s mountain). With its strategic position on the river Pregel, it became a trading center and an important medieval city. The river flowed around the island of Kneiphof (literally, pub yard) and divided the city into four regions connected by seven bridges: Blacksmith’s bridge, Connecting bridge, High bridge, Green bridge, Honey bridge, Merchant’s bridge, and Wooden bridge: Figure 3 shows a seventeenth-century map of the city. Königsberg later became the capital of East Prussia and more recently became the Russian city of Kaliningrad, while the river Pregel was renamed Pregolya.

In 1727 Leonhard Euler began working at the Academy of Sciences in St Petersburg. He presented a paper to his colleagues on 26 August 1735 on the solution of ‘a problem relating to the geometry of position’: this was the Königsberg bridges problem. He also addressed the generalized problem: given any division of a river into branches and any arrangement of bridges, is there a general method for determining whether such a route exists?

In 1736 Euler wrote up his solution in his celebrated paper in the Commentarii Academiae Scientiarum Imperialis Petropolitanae under the title ‘Solutio problematis ad geometriam situs pertinentis’ [2], numbered E53 in the Eneström index. Euler’s diagram of the Königsberg bridges appears in Figure 4. Although dated 1736, Euler’s paper was not actually published
until 1741, and was later reprinted in the new edition of the Commentarii (Novi Commentarii . . . ) which appeared in 1752.

A full English translation of this paper appears in several places—for example, in [1] and [6]. The paper begins:

1. In addition to that branch of geometry which is concerned with distances, and which has always received the greatest attention, there is another branch, hitherto almost unknown, which Leibniz first mentioned, calling it the geometry of position \([\text{Geometriam situs}]\). This branch is concerned only with the determination of position and its properties; it does not involve distances, nor calculations made with them. It has not yet been satisfactorily determined what kinds of problem are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position—especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this problem, as an example of the geometry of position.

2. The problem, which I am told is widely known, is as follows: in Kônigsberg . . .

This reference to Leibniz and the geometry of position dates back to 8 September 1679, when the mathematician and philosopher Gottfried Wilhelm Leibniz wrote to Christiaan Huygens as follows [5]:

I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently,
in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes . . .

Leibniz introduced the term *analysis situs* (or *geometria situs*), meaning the analysis of situation or position, to introduce this new area of study. Although it is sometimes claimed that Leibniz had vector analysis in mind when he coined this phrase (see, for example, [8] and [11]), it was widely interpreted by his eighteenth-century followers as referring to topics that we now consider ‘topological’—that is, geometrical in nature, but with no reference to metrical ideas such as distance, length or angle.

3. Euler’s Königsberg letters

It is not known how Euler became aware of the Königsberg bridges problem. However, as we shall see, three letters from the Archive Collection of the Academy of Sciences in St Petersburg [3] shed some light on his interest in the problem (see also [10]).

Carl Leonhard Gottlieb Ehler was the mayor of Danzig in Prussia (now Gdańsk in Poland), some 80 miles west of Königsberg. He corresponded with Euler from 1735 to 1742, acting as intermediary for Heinrich Kühn, a local mathematics professor. Their initial communication has not been recovered, but a letter of 9 March 1736 indicates they had discussed the problem and its relation to the ‘calculus of position’:

You would render to me and our friend Kühn a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position [*Calculi Situs*], worthy of your great genius. I have added a sketch of the said bridges . . .

Euler replied to Ehler on 3 April 1736, outlining more clearly his own attitude to the problem and its solution:

... Thus you see, most noble Sir, how this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime, most noble Sir, you have assigned this question to the geometry of position, but I am igno-
rant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed in this way . . .

Around the same time, on 13 March 1736, Euler wrote to Giovanni Mari
noni, an Italian mathematician and engineer who lived in Vienna and was Court Astronomer in the court of Kaiser Leopold I. He introduced the problem as follows (see Figure 6):

A problem was posed to me about an island in the city of Königsberg, surrounded by a river spanned by seven bridges, and I was asked whether someone could traverse the separate bridges in a connected walk in such a way that each bridge is crossed only once. I was informed that hitherto no-one had demonstrated the possibility of doing this, or shown that it is impossible. This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position [geometriam Situs], which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not . . .
4. Euler’s 1736 paper

Euler’s paper is divided into twenty-one numbered paragraphs, of which the first ascribes the problem to the geometry of position as we saw above, the next eight are devoted to the solution of the Königsberg bridges problem itself, and the remainder are concerned with the general problem. More specifically, paragraphs 2–21 deal with the following topics (see also [12]):

Fig. 6. Euler’s letter to Marinoni

**Paragraph 2.** Euler described the problem of the Königsberg bridges and its generalization: ‘whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?’
Paragraph 3. In principle, the original problem could be solved exhaustively by checking all possible paths, but Euler dismissed this as ‘laborious’ and impossible for configurations with more bridges.

Paragraphs 4–7. The first simplification is to record paths by the land regions rather than bridges. Using the notation in Figure 4, going south from Kneiphof would be notated $AB$ whether one used the Green Bridge or the Blacksmith’s Bridge. The final path notation will need to include an adjacent $A$ and $B$ twice; the particular assignment of bridges $a$ and $b$ is irrelevant. A path signified by $n$ letters corresponds to crossing $n - 1$ bridges, so a solution to the Königsberg problem requires an eight-letter path with two adjacent $A/B$ pairs, two adjacent $A/C$ pairs, one adjacent $A/D$ pair, etc.

Paragraph 8. What is the relation between the number of bridges connecting a land mass and the number of times the corresponding letter occurs in the path? Euler developed the answer from a simpler example (see Figure 7). If there is an odd number $k$ of bridges, then the letter must appear $(k + 1)/2$ times.

Fig. 7. A simple case

Paragraph 9. This is enough to establish the impossibility of the desired Königsberg tour. Since Kneiphof is connected by five bridges, the path must contain three $A$s. Similarly, there must be two $B$s, two $C$s, and two $D$s. In Paragraph 14, Euler records these data in a table.

<table>
<thead>
<tr>
<th>region</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bridges</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>frequency</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Summing the final row gives nine required letters, but a path using each of the seven bridges exactly once can have only eight letters. Thus there can be no Königsberg tour.

Paragraphs 10–12. Euler continued his analysis from Paragraph 8: if there is an even number $k$ of bridges connecting a land mass, then the corresponding letter appears $k/2 + 1$ times if the path begins in that region, and $k/2$ times otherwise.
Paragraphs 13–15. The general problem can now be addressed. To illustrate the method Euler constructed an example with two islands, four rivers, and fifteen bridges (see Figure 8).

This system has the following table, where an asterisk indicates a region with an even number of bridges.

<table>
<thead>
<tr>
<th>region</th>
<th>A*</th>
<th>B*</th>
<th>C*</th>
<th>D</th>
<th>E</th>
<th>F*</th>
</tr>
</thead>
<tbody>
<tr>
<td>bridges</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>frequency</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

The frequencies of the letters in a successful path are determined by the rules for even and odd numbers of bridges, developed above. Since there can be only one initial region, he records \(k/2\) for the asterisked regions. If the frequency sum is one less than the required number of letters, there is a path using each bridge exactly once that begins in an asterisked region. If the frequency sum equals the required number of letters, there is a path that begins in an unasterisked region. This latter possibility is the case here: the frequency sum is 16, exactly the number of letters required for a path using 15 bridges. Euler exhibited a particular path, including the bridges:

\[E \ a \ F \ b \ B \ c \ F \ d \ A \ e \ F \ f \ C \ g \ A \ h \ C \ i \ D \ k \ A \ m \ E \ n \ A \ p \ B \ o \ E \ l \ D.\]

Paragraph 16–19. Euler continued with a simpler technique, observing that:

... the number of bridges written next to the letters \(A, B, C\), etc. together add up to twice the total number of bridges. The reason for this
is that, in the calculation where every bridge leading to a given area is counted, each bridge is counted twice, once for each of the two areas which it joins.

This is the earliest version known of what is now called the *handshaking lemma*. It follows that in the bridge sum, there must be an even number of odd summands.

Paragraph 20. Euler stated his main conclusions:

If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these two areas.

If, finally, there are no areas to which an odd number of bridges lead, then the required journey can be accomplished starting from any area.

Paragraph 21. Euler concluded by saying:

When it has been determined that such a journey can be made, one still has to find how it should be arranged. For this I use the following rule: let those pairs of bridges which lead from one area to another be mentally removed, thereby considerably reducing the number of bridges; it is then an easy task to construct the required route across the remaining bridges, and the bridges which have been removed will not significantly alter the route found, as will become clear after a little thought. I do not therefore think it worthwhile to give any further details concerning the finding of the routes.

Note that this final paragraph does not prove the existence of a journey when one is possible, apparently because Euler did not consider it necessary. So Euler provided a rigorous proof only for the first of the three conclusions. The first satisfactory proof of the other two results did not appear until 1871, in a posthumous paper by Carl Hierholzer (see [1] and [4]).

5. The modern solution

The approach mentioned in the first section developed through diagram-tracing puzzles discussed by Louis Poinsot [7] and others in the early-nineteenth century. The object is to determine whether a figure can be drawn with a single stroke of the pen in such a way that no edge is repeated. Considering the figure to be drawn as a graph, the general conditions in Paragraph 20 take the following form:
If there are more than two vertices of odd degree, then such a drawing is impossible.

If, however, exactly two vertices have odd degree, then the drawing is possible if it starts with either of these two vertices.

If, finally, there are no vertices of odd degree, then the required drawing can be accomplished starting from any vertex.

So the 4-vertex graph shown in Figure 2, with one vertex of degree 5 and three vertices of degree 3, cannot be drawn with a single stroke of the pen so that no edge is repeated. In contemporary terminology, we say that this graph is not Eulerian. The arrangement of bridges in Figure 8 can be similarly represented by the graph in Figure 9, with six vertices and fifteen edges. Exactly two vertices ($E$ and $D$) have odd degree, so there is a drawing that starts at $E$ and ends at $D$, as we saw above. This is sometimes called an Eulerian trail.

![Fig. 9. The graph of the bridges in Figure 8](image)

However, it was some time until the connection was made between Euler’s work and diagram-tracing puzzles. The ‘Königsberg graph’ of Figure 2 made its first appearance in W. W. Rouse Ball’s *Mathematical Recreations and Problems of Past and Present Times* [9] in 1892.

Background information, including English translations of the papers of Euler [2] and Hierholzer [4], can be found in [1]; an English translation of Euler’s paper also appears in [6].
References

On November 14, 1750 Leonhard Euler sent a letter from Berlin to his friend Christian Goldbach in St. Petersburg announcing his discovery of a simple relationship between the features on a polyhedron [19, p. 332–3]. This observation, now known as Euler’s polyhedral formula, is one of the most beloved theorems in mathematics. (A 1990 survey of mathematicians found the polyhedral formula to be the second most beautiful theorem in all of mathematics [40].) Euler stated the theorem as follows [11]¹.

THEOREM: In every solid enclosed by plane faces, the number of faces along with the number of solid angles exceeds the number of edges by two. This relationship is typically expressed as \( F - E + V = 2 \) where \( F \), \( E \), and \( V \) denote the number of faces, edges, and vertices of a polyhedron.

Euler wrote two papers on the polyhedral formula, both published in 1758. The first paper, written in 1750, contains the statement of the theorem [10], and the second, written the following year, contains his proof [11] (henceforth we shall refer to them by their Eneström index numbers, E230 and E231, respectively). Euler wrote these two papers because he was interested in classifying all polyhedra. He wanted to develop the theory of stereometry (solid geometry) just as it had been developed for planimetry (planar geometry). He did not achieve his goal of classifying all polyhedra, and he never returned to this topic after publishing these two papers.

Euler’s seemingly elementary observation proved to be an important theorem in mathematics that was generalized in many directions. The ideas

¹ A full English translation of [11] can be found at [12].
contained in Euler’s formula were later extended to polyhedra with non-trivial topology, polyhedra in higher dimensions, planar graphs, topological of surfaces, other topological spaces, and abstract algebraic entities. From these generalizations countless applications were found.

In this paper we present Euler’s proof of the polyhedral formula. We look closely at his hypotheses and his proof, discuss the flaw in his argument and show how it can be repaired. We also present the related work of other mathematicians prior to 1850. It is during this period that the theory for polyhedra develops, whereas after 1850 the focus becomes much more topological.

1. The polyhedral formula

In his letter to Goldbach, Euler wrote, “It astonishes me that these general properties of stereometry have not, as far as I know, been noticed by anyone else” [19]. In 1750 all of the accumulated knowledge about polyhedra was metric. There were many theorems and formulas about volume, surface area, angle measures, inscribability, etc. No one prior to Euler (except, as we will see, Descartes) looked at polyhedra with an eye toward their combinatorial properties.

It was not only the formula that went unnoticed prior to 1750. In this same letter Euler described “the junctures where two faces come together along their sides, which, for lack of an accepted term, I call acies” [19]. Until he gave them a name, no one had explicitly referred to the edges of a polyhedron. Acies is a Latin term which is commonly used for the sharp edge of a weapon, a beam of light, or an army lined up for battle. Giving a name to this feature may seem to be a trivial point, but it is not. It is a crucial observation that the edge of a polyhedron is an important feature to count. For the faces of a polyhedron Euler uses the well established Latin term hedra, which translates to face or base. He refers to the vertices of a polyhedron as angulus solidus, or solid angles.

It is clear that Euler understood the importance of these three features. In E230 he wrote [10]:

Therefore three kinds of bounds are to be considered in any solid body; namely 1) points, 2) lines and 3) surfaces, or, with the names specially used for this purpose: 1) solid angles, 2) edges and 3) faces. These three kinds of bounds completely determine the solid.

Viewed in this light we see that Euler’s formula is a way of relating objects of different dimensions – the zero-dimensional vertices, one-dimensional edges, and two-dimensional faces. This theorem and Euler’s 1736 solution to
the Bridges of Königsberg problem [9] were among the earliest contributions to the young field of \textit{analysis situs}, or topology.

In E230 Euler begins his study of stereometry. In this paper he states the polyhedral formula, and he verifies that it holds for a variety of polyhedra, but he is unable to give a proof. In E231 he recalls:

After the consideration of many types of solids I came to the point where I understood that the properties which I had perceived in them clearly extended to all solids, even if it was not possible for me to show this in a rigorous proof. Thus, I thought that those properties should be included in that class of truths which we can, at any rate, acknowledge, but which it is not possible to prove.

Then in E231 he gives a proof. The idea of the proof is to cut away vertices, one at a time, until four vertices remain. This excision is done in such a way that $F - E + V$ remains unchanged at each step. At the end the resulting polyhedron is a triangular pyramid which satisfies the polyhedral formula. Thus the original polyhedron does as well. We now give Euler’s argument in more detail.

Begin with a polyhedron $P$ having $F$ faces, $E$ edges, and $V$ vertices. Choose any vertex $O$ of $P$. We must remove $O$ in such a way that that the resulting polyhedron $P'$ has $V - 1$ vertices. $O$ can be any vertex of $P$, but if $P$ is a pyramid we may wish to avoid choosing $O$ as the apex, for in this case $P'$ will collapse into a polygon (although Euler remarks that a polygon, thought of as a polyhedron with two faces, still satisfies the formula).

Remove $O$ by cutting away triangular pyramids. For each excised pyramid one vertex is $O$ and the other three are vertices adjacent to $O$ in $P$. A simple case is shown in Figure 1.

![Fig. 1. Removing the vertex $O$ by cutting away pyramids](image)

More specifically, if the degree of $O$ is $n$, then the $n$ vertices adjacent to $O$ form a (perhaps nonplanar) $n$-gon. By adding $n - 3$ diagonals we triangulate this polygon into $n - 2$ triangles, $T_1, \ldots, T_{n-2}$. The $n - 2$ pyramids we cut away have the triangles $T_i$ as bases and the vertex $O$ as the
apex. Notice that since there are several ways to triangulate a polygon, this decomposition is not unique.

We must now determine the number of faces and edges in $P'$. First we make two simplifying assumptions: that all of the faces of $P$ meeting at $O$ are triangular, and that no pair of neighboring triangles $T_i$ and $T_{i+1}$ are coplanar (the polyhedron in Figure 1 has both of these properties). In this case, we cut away the $n$ faces meeting at $O$ and added back the $n - 2$ triangles, $T_1, \ldots, T_{n-2}$. Likewise we cut away the $n$ edges meeting at $O$ and added back the $n - 3$ edges between the $T_i$. Thus, $P'$ has $F - n + (n - 2) = F - 2$ faces and $E - n + (n - 3) = E - 3$ edges.

Now, consider the case that there are $\nu$ nontriangular faces meeting at $O$. When the triangular pyramids are removed they cut through these $\nu$ faces, and in each case leave behind part of a face and create a new edge (see Figure 2). So, we remove $n$ faces and add back $n - 2 + \nu$, and we remove $n$ edges and add back $n - 3 + \nu$. Thus $P'$ has $F - 2 + \nu$ faces and $E - 3 + \nu$ edges.

![Fig. 2. When this vertex is removed we find $\nu = 1$ (middle) and $\mu = 1$ (right).](image)

Suppose that among the triangles $T_1, \ldots, T_{n-2}$ on $P'$ there are $\mu$ pairs of coplanar neighbors (see Figure 2). Each pair of such neighbors merge to form a single face, thus we lose one edge and one face. So, in total we add back $\mu$ fewer faces and $\mu$ fewer edges. Thus, $P'$ has $F - 2 + \nu - \mu$ faces and $E - 3 + \nu - \mu$ edges.

Although the numbers of edges and faces may go up or down when a vertex is removed, the difference between the number of edges and the number of faces decreases by one,

$$(E - 3 + \nu - \mu) - (F - 2 + \nu - \mu) = E - F - 1.$$ 

Continue cutting away vertices in this way, removing $n$ in total, until only 4 remain. Thus we obtain a triangular pyramid (with 4 faces and 6 edges). The difference in the number of edges and faces is $E - F - n = 6 - 4 = 2$ and the number of vertices is $V - n = 4$. Solving for $n$ and substituting we have $E - F - (V - 4) = 2$, or $F - E + V = 2$. 

Although we omit the proof here, Euler uses this same technique to prove a second theorem, that the sum of all the plane angles of a polyhedron is $2\pi(V - 2)$ (a plane angle is an angle in the polygon forming a face of the polyhedron). In E230 Euler proved that this theorem is equivalent to the polyhedral formula. He was the first mathematician to publish the angle sum formula, but, as we will see, it was known to Descartes.

2. The flaw and the repair

In 1924 Henri Lebesgue pointed out that Euler was not sufficiently careful when he gave his proof of the polyhedral formula [22]. The first problem is that he never defines the objects he is studying. The second problem is that he is too casual when describing the decomposition process. As we will see, Euler’s proof fails for both convex and for nonconvex polyhedra. However, in the case of convex polyhedra, Euler’s proof can be salvaged.

Euler does not use the word polyhedron. Instead he refers to “solids enclosed by plane faces” (solida hedris planis inclusa). We could take this phrase to be synonymous with polyhedron, but in 1750 there was no explicitly-stated definition of polyhedron either. As Poincaré wrote, “the objects occupying mathematicians were long ill defined; we thought we knew them because we represented them with the senses or the imagination; but we had of them only a rough image and not a precise concept upon which reasoning could take hold” [30]. It is reasonable to believe that Euler, like the Greeks, made the unstated assumption that every polyhedron is convex. It was not until the nineteenth century that mathematicians attempted to formulate a precise definition. One should consult Lakatos’ excellent book [21] for an extended discussion of the many attempts to define polyhedron.

Convexity is important for Euler’s decomposition algorithm. It may be impossible to cut away a vertex when the polyhedron is not convex in its vicinity (such as the vertices around the waist of the hourglass in Figure 3). It may be impossible to remove a locally convex vertex when the polyhedron is nonconvex (such as the apex of the polyhedron in the center). As a worst-case scenario it may be impossible to remove any single vertex using Euler’s method. In the third polyhedron in Figure 3 the vertices located in the indentations cannot be removed at all, and when the vertex located at the center of a star is removed the number of vertices decreases by six, not by one.

Problems may arise for convex polyhedra as well. Euler does not give instructions for how to decompose a polyhedron. Instead he presents a few
examples and describes how to decompose these polyhedra. The following example shows that after a vertex is removed, a convex polyhedron may become nonconvex. Here, the vertex to be removed, $O$, has four adjacent vertices $A$, $B$, $C$, and $D$. He writes:

This can be done in two ways (Fig. 3 [our Figure 4]): two pyramids will have to be cut away, either $OABC$ and $OACD$ or $OABD$ and $OBCD$. And if points $A, B, C, D$ are not in the same plane the resulting solids will have a different shape accordingly.

It is not difficult to see that if $A, B, C, D$ are not coplanar, then one of resulting solids will not be convex. He does not acknowledge that one decomposition is acceptable and the other is not. This example shows that Euler was not concerned, or not aware of issues of convexity when drafting his proof.

Worse still, Lebesgue showed that it is possible to apply Euler’s algorithm to a convex polyhedron and obtain a degenerate polyhedron that fails to satisfy the polyhedral formula. In Figure 5 we see that one choice yields a polyhedron while the other choice yields two polyhedra joined along an edge. Similarly, we may obtain two polyhedra joined at a vertex or two disjoint polyhedra (see Figure 6). None of these polyhedra are topological balls. However, like the vertex $O$ in Figure 5, the vertices labeled $O$ in
Fig. 5. Euler’s technique, applied to the polyhedron on the left may (middle) or may not (right) produce a polyhedron.

Figure 6 can be removed in such a way that the resulting polyhedron is convex.

Fig. 6. More degenerate polyhedra

Indeed, as Samelson shows in [33], given any convex polyhedron we can decompose it in the way Euler intended. The only stipulation is that at each stage in the decomposition, the pyramids must be chosen strategically, not arbitrarily. Essentially, this amounts to finding a triangulation $T_1, \ldots, T_{n-2}$ that preserves the convexity of the polyhedron.

Recall that the convex hull of a set is the smallest convex set containing this set. It is easy to see that a polyhedron is convex if and only if it is the convex hull of its vertex set. Thus, if we take the vertex set for $P$, remove the vertex $O$, and then take the convex hull, we obtain a convex polyhedron $P'$. Doing so creates a convex cap in place of the removed vertex. This cap may produce the desired triangulation $T_1, \ldots, T_{n-2}$, but in some cases (corresponding to $\mu \neq 0$) some of the new faces may have more than three sides. These faces may be triangulated arbitrarily. Notice that it is only in this case that the choices can be made in the removal of $O$. Even in this case the resulting polyhedron $P'$ is unique.
3. Legendre’s proof

The first rigorous proof of the polyhedral formula was given by Adrien-Marie Legendre in 1794. The proof appeared in the first edition of his popular textbook *Éléments de Géométrie* [23]. His elegant proof is not a reworking of Euler’s proof, but presents a completely new and unexpected approach. The proof is not a combinatorial proof, but instead it uses metric properties of spheres.

The key ingredient in the proof is a theorem proved independently by Thomas Harriot in 1603 [28] (he did not publish the result) and Albert Girard in 1629 [16]. They showed that a geodesic triangle on a sphere of radius \( r \) with interior angles \( a, b \), and \( c \) has area

\[
A = r^2(a + b + c - \pi).
\]

More generally, a geodesic polygon with interior angles \( a_1, a_2, \ldots, a_n \) has area

\[
A = r^2(a_1 + \ldots a_n - (n - 2)\pi).
\]

To prove Euler’s formula, place the polyhedron inside a sphere (which we assume to be the unit sphere) and project the edges and vertices onto the sphere from the sphere’s center. In this way the faces of the polyhedron project to geodesic polygons. The sphere has area \( 4\pi \), but the area can also be computed by summing the areas of the \( F \) geodesic polygons. By the Harriot-Girard theorem, the area is

\[
4\pi = \sum_{i=1}^{I} a_i + \sum_{j=1}^{F} (n_j - 2)\pi = \sum_{i=1}^{I} a_i + \pi \sum_{j=1}^{F} n_j - 2\pi F.
\]

where the first sum is taken over all interior angles of all of the geodesic polygons. Since the sum of the interior angles that meet at a vertex is \( 2\pi \) we have \( \Sigma a_i = 2\pi V \). Since each edge borders two faces \( \pi \Sigma n_j = 2\pi E \). Thus we obtain

\[
4\pi = 2\pi F - 2\pi E + 2\pi V.
\]

Dividing by \( 2\pi \) we obtain Euler’s formula.

Legendre, like Euler, assumed his polyhedron was convex. That way we can take any point inside the polyhedron to be the center of the sphere. However, in 1810 in the appendix to [31] Louis Poinsot remarked that Legendre’s proof applies without alteration to any polyhedron that has such a central point from which the projection can be made (so-called star-convex polyhedra). Thus, Poinsot was the first person to explicitly show that some nonconvex polyhedra satisfy Euler’s formula.
4. The exceptions of Lhuilier, Hessel, and Poinsot

At the beginning of the nineteenth century mathematicians were trying to come to grips with Euler’s formula. They wanted to determine exactly which polyhedra satisfied Euler’s formula, or using the terminology of Johann Friedrich Christian Hessel, which polyhedra were Eulerian.

Some stated the polyhedral formula only for convex polyhedra, not knowing or not caring that it held more generally. E. de Jonquières wrote that, “in invoking Legendre, and like high authorities, one only fosters a widely spread prejudice that has captured even some of the best intellects: that the domain of validity of the Euler theorem consists only of convex polyhedra” [7]. For, as D. M. Y. Sommerville writes, “convexity is to a certain extent accidental, and a convex polyhedron might be transformed, for example, by a dent or by pushing in one or more of the vertices, into a nonconvex polyhedron with the same configurational numbers” [36]. Others erred in the other extreme by stating that it applied to all polyhedra.

The first few decades of the nineteenth century saw several examples of non-Eulerian polyhedra. In 1811 the Swiss mathematician Simon-Antoine-Jean Lhuilier wrote a long paper on polyhedra [24] and submitted a memoir to Joseph Diaz Gergonne’s journal Annales de Mathématiques, but it was too long to print. (It is amusing to note that l’huilier means “the oilcan” or “the one who oils,” thus Lhuilier may be called “The Oiler.”) In 1813 Gergonne published his own shortened account of Lhuilier’s paper [25] and included in it ideas of his own.

In this memoir Lhuilier gives three classes of counterexamples to Euler’s formula, or exceptions as he called them. An example of each type of exception is shown in Figure 7. The first polyhedron has a face that is not simply connected—it is topologically an annulus. Lhuilier remarked that every “inner polygon” within a face would increase by one the quantity $F - E + V$. The second polyhedron has the shape of a polyhedral torus. Lhuilier observed that each “tunnel” decreased the alternating sum by two. Finally, the third example is a cube with a cube-shaped cavity in the interior. This exception was inspired by a mineral in the collection of his friend Professor Pictet that had a colored crystal suspended inside a clear crystal (In 1832 Hessel also found such a crystal—in his case he identified it as a lead sulphide cube within a calcium chloride crystal [17].) Each cavity increases the alternating sum by two.

Thus Lhuilier proposed a modified version of the polyhedral formula. A polyhedron with $T$ tunnels, $C$ cavities, and $P$ inner polygons satisfies

$$F - E + V = 2 - 2T + P + 2C.$$  

(This is the earliest incarnation of the topological theorem that the Euler
characteristic of a topological surface of genus $g$ is $2 - 2g$)

In his account of Lhuilier’s work Gergonne wrote, “one will easily be convinced that Euler’s Theorem is true in general for all polyhedra, whether they are convex or not, except for those instances that will be specified” [25]. However, there are exceptions that do not fit comfortably into Lhuilier’s three classes. In Figure 8 we see a polyhedron with a face possessing two inner polygons that share a common vertex; a polyhedron with a branched tunnel in it; a polyhedron with a torus-shaped cavity; and a polyhedral torus without an obvious tunnel.

In 1832 Hessel, a mineralogist who is most well-known for his mathematical investigation of symmetry classes of minerals [2], presented five exceptions to the polyhedra formula [17]. Shortly after submitting the paper he learned of Lhuilier’s memoir from two decades earlier and discovered that three of his five exceptions coincided with Lhuilier’s. Hessel believed that many people were unaware of these important exceptions, so he decided not to withdraw the publication [21]. His two new exceptions are shown in Figure 9. One is a polyhedron formed from two polyhedra joined along an edge and the other is a polyhedra formed from two polyhedra joined at a vertex. It is debatable whether these figures should be classified as polyhedra, but they certainly fail to satisfy the polyhedral formula.

In 1810 Poinsot wrote about the four star polyhedra shown in Figure 10 [31]. Unbeknownst to Poinsot, two of the four star polyhedra, the great and small stellated dodecahedra, can be found in Kepler’s Harmonice Mundi from 1619 [20], and prior to that they appeared in artwork by Wentzel.
Jamnitzer and Paolo Uccello, respectively. Poinsot was the first to present the other two polyhedra, the great dodecahedron and the great icosahedron, in a mathematical context, although the former is also seen in the artwork of Jamnitzer. This collection of four polyhedra is now referred to as the Kepler-Poinsot polyhedra.

The Kepler-Poinsot polyhedra are not exceptions to Euler’s formula if they are viewed as nonconvex polyhedra formed from triangular faces. However, both Kepler and Poinsot imagined that these polyhedra had self-intersecting faces and were in fact regular. For instance, when we view the great dodecahedron as a polyhedron with 12 pentagonal faces it does not satisfy the polyhedral formula (it has 30 edges and 12 vertices, thus $12 - 30 + 12 = -6$). The other three polyhedra are also exceptions. We now know that the Kepler-Poinsot polyhedra do not obey the polyhedral formula because they are not topological spheres.

5. Cauchy’s proof

The first two of Augustin Louis Cauchy’s many mathematical papers concerned the theory of polyhedra. They were completed in 1811 and 1812 while he was a engineer working at the harbor of Cherbourg, before he began his mathematical career. He proved that the four Kepler-Poinsot
polyhedra were unique [3]. He proved his famous rigidity theorem—a convex polyhedron is completely determined by its faces [4]. He also gave a new proof of Euler’s polyhedral formula and extended it in several new, important directions [3]. Both papers appeared in 1813.

The first notable feature that distinguishes Cauchy’s proof from Euler’s and Legendre’s is that it applies to polyhedra that are hollow, not solid. Despite what some historians contend, Cauchy still viewed polyhedra as solid, but his proof used the “convex surface of a polyhedron.”

In Cauchy’s proof we begin by choosing a face, and then “by transporting onto this face all the other vertices without changing their number, one will obtain a plane figure made up of several polygons contained in a given contour” [3] (Figure 11).

In 1813 Gergonne describes this process as follows: “Take a polyhedron, one of its faces being transparent; and imagine that the eye approaches this face from the outside so closely that it can perceive the inside of all the other faces; this is always possible when the polyhedron is convex. The things being so arranged, let us imagine that on the plane of the transparent face a perspective is made of the set of all the others” [25]. Lakatos puts a modern spin on Gergonne’s idea by suggesting that a camera be placed near the removed face, then the network will appear on the photographic print [21].

Thus, Cauchy realized that it is sufficient to relate numbers of faces, edges, and vertices in this planar network of polygons, or what we would today call a planar graph or map. Cauchy proved that every such graph satisfies \( F - E + V = 1 \). Then it is easy to complete the proof of the polyhedral formula since the graph has the same number of edges and vertices as the polyhedron and it has one fewer face.

Cauchy begins his proof by triangulating the graph (see Figure 12). He argues that doing so not change the quantity \( F - E + V \). Then, “we remove successively the various triangles, so that only one remains in the end, starting with those that border the external contour, and then removing only those which, by earlier reductions, have one or two sides belonging to that contour” [3]. In one case the triangle can be removed by taking away
The Polyhedral Formula

one edge and no vertices (such as the removal of triangle number 1), and in the other case the triangle can be eliminated by removing two edges and a vertex (such as the removal of triangle number 2). In either case, the quantity \(F - E + V\) remains unchanged. Thus, since \(F - E + V = 1\) for the final triangle, \(F - E + V = 1\) for the original graph.

Cauchy's proof was later criticized. Just as Euler ran into trouble by failing to give explicit instructions on what order to remove the pyramids, Cauchy does not give instructions on how to cut away the triangles. If we are not careful it is possible to follow Cauchy's algorithm and obtain a disconnected graph, for which relation fails to hold (see Figure 13).

Freudenthal writes the following about: “In nearly all cases he left the final form of his discoveries to the next generation. In all that Cauchy achieved there is an unusual lack of profundity. . . He was the most superficial of the great mathematicians, the one who had a sure feeling for what was simple and fundamental without realizing it” [15]. Cauchy's proof of the polyhedral formula is an apt example of this. The proof applies to very broad class of polyhedra. Using the language of today, the polyhedron must be a topological sphere and have simply connected faces. These properties are guaranteed by convexity, but convexity is by no means necessary. In the statement of his theorem he omits the word convex, giving the impression that he realized the power of the proof. However, in the proof he explicitly states that he is considering convex polyhedra. He never addresses this inconsistency. Some historians, such as Steinitz [38] and Lakatos [21], claim that Cauchy knew his proof applied to some or perhaps all nonconvex polyhedra but this is not clear from what he wrote.
Regardless of whether he recognized that the result could be extended easily to some nonconvex polyhedra, it was quickly seen by others. In 1813, the same year that Cauchy’s paper was published, Gergonne gave his own proof of Euler’s formula. Afterward he wrote, “one might prefer still, with reason, the beautiful proof of Mr. Cauchy, who has the precious advantage of not assuming that the polyhedron is convex” [25].

Just like Cauchy did not recognize the full strength of his proof for polyhedra, he also did not see the full strength of his theorem for graphs. This theorem was generalized by Cayley in 1861 [5] who showed that it applies to graphs with curved edges (this fact was noticed independently by Listing in 1861 [27] and Jordan in 1866 [18]). Moreover, Cauchy proved the theorem for any collection of polygons contained in a polygonal outline. We now know that it applies to any connected planar graph.

In this same paper Cauchy gives a glimpse of the higher-dimensional generalization of Euler’s formula. He proves that if faces, edges, and vertices are inserted into the interior of a convex polyhedron dividing it into \( P \) convex polyhedra and if the total number of faces, edges, and vertices (including those in the interior) is \( F, E, \) and \( V \), then they satisfy \(-P + F - E + V = 1\). This equality shows that the Euler characteristic of the 3-ball is 1. In 1855 Schlafli generalized Cauchy’s result to polytopes (as they are now called) of all dimensions. [35].

6. Von Staudt’s proof

The first half of the nineteenth century saw several exceptions to Euler’s formula and many new proofs. We will not give an account of all of the proofs here (see e.g., [25,37]). All of the proofs that appeared before 1847 apply comfortably to convex polyhedra, and in some cases they can be extended to a broader class of polyhedra. However, no one had yet given a broad classification of Eulerian polyhedra. It was in this year that Georg Christian von Staudt, in his book Geometrie der Lage [39], finally gave a very general set of criteria that describe Eulerian polyhedra. Von Staudt’s criteria for the polyhedra, which he assumed were hollow, are:

(i) It is possible to get from any vertex to any other vertex by a path of edges.

(ii) Every simple closed path of edges divides the polyhedron into two components.

He then gave a beautiful argument that proved that any polyhedron satisfying these hypotheses is Eulerian. We now give a brief sketch of von Staudt’s proof (using modern terminology).
Create a spanning tree for the edges of the polyhedron. Such a tree is shown in second image in Figure 14 as a thick solid line. This tree has $V$ vertices. By property (1) the tree is connected, thus it contains $V - 1$ edges.

Now, place a new vertex inside each face. Draw a dashed edge from one face to an adjacent face whenever the two faces are not separated by an edge of the first tree. Property (ii) implies that this graph is connected. Moreover, this graph is a tree, for if it contained a circuit, then by property (ii) the original path would not be connected. Since this tree has $F$ vertices, it has $F - 1$ edges. Every edge in the original polyhedron is either in the original spanning tree or is crossed by a dashed edge. Thus the number of edges of the polyhedron is:

$$E = (V - 1) + (F - 1).$$

Rearranging terms we obtain $F - E + V = 2$.

7. Prehistory of the polyhedral formula: Descartes’ lost notes

By 1860 the polyhedral formula was well-known and it had Euler’s name firmly attached to it. It was in this year that Foucher de Careil discovered a note hand-written by Gottfried Leibniz indicating that René Descartes knew the polyhedral formula in approximately 1630, 120 years before Euler’s proof.

The story of the document now called De Solidorum Elementis is fascinating and unlikely. Descartes died in Sweden in 1650 while visiting Queen Christina. After his death his personal effects were shipped back to Paris, but they were nearly lost when the boat wrecked in the Seine. After his unpublished manuscripts were hung to dry, they were made available for public inspection. During a visit to Paris in 1675-6 Leibniz copied some of Descartes’ notes, including De Solidorum Elementis. Descartes’ document was never seen again and Leibniz’s copy was lost until its discovery in a
dusty cupboard of the Royal Library of Hanover in 1860. (For more details see [13].)

This document contains Descartes’ observations on polyhedra. It has the angle sum formula that appeared in Euler’s E230 and E231. It also has a relation between the numbers of plane angles, faces, and vertices ($P$, $F$, and $V$, respectively), $P = 2F + 2V - 4$. As Euler proved, the first formula is equivalent to the polyhedral formula, and second can be transformed into the polyhedral formula by substituting $P = 2E$.

Some historians contend that Descartes’ knowledge of these relations entitles him to credit for discovering the polyhedral formula. As de Jonquières wrote, “It cannot be denied then that he knew it, since it is a deduction so direct and so simple, we say so intuitive, from the two theorems that he had just stated” [13]. Today we frequently encounter the polyhedral formula called the Descartes-Euler formula.

Other historians point to the importance of edges in the polyhedral formula. They argue that polyhedral formula is a theorem about dimension—that it must relate the numbers of cells of 0, 1, and 2 dimensions. There is no indication that Descartes viewed polyhedra in this way. Lebesgue, after carefully examining the manuscript, wrote, “Descartes did not enunciate the theorem; he did not see it” [22].

8. After 1850

As Pont wrote, “After one hundred years of history (1750–1850), the theorem of Euler traversed the various stages allocated to an honest theorem: empirical appearance, approximate statement, proof in a particular case, exact statement, generalization” [32]. However, during this time no one noticed the topological significance of Euler’s formula. This observation came in 1861 in a long work by Johann Benedict Listing, a student of Gauss [27]. Listing is known as one of the early pioneers of topology. We can thank him for coining the term “topology” in the title of his 1847 Vorstudien zur topologie [26] and for co-discovering the Möbius strip (along with Möbius).

In the second half of the nineteenth century the mathematical subject called topology began to take shape with contributions from Jordan, Riemann, Möbius, Klein, Betti, Dyck, and others. In a series of papers starting in 1895 Poincaré unveiled the blueprint for the field of algebraic topology and gave the first truly modern interpretation of Euler’s formula [29]. The alternating sum, now referred to as Euler characteristic (or the Euler-Poincaré characteristic), is one of the most fundamental topological invariants. The myriad applications of the Euler characteristic are far too
numerous to list.

In order to appreciate the current state of algebraic topology it is important to recognize the important contributions of Euler and the other mathematicians who studied polyhedra from 1750 to 1850.

References


On the Recognition of Euler among the French, 1790-1830

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1. The rise of Paris to mathematical eminence

When Jean d’Alembert, Daniel Bernoulli and Leonhard Euler died within 18 months of each other in 1782 and 1783, a generation of researchers passed and the inheritance came to their principal successors. This change also established Paris as by far the main mathematical centre, a status that it was to enjoy until well into the 1820s. The initial figures, some already well recognized, were C. Bossut (1730-1814), G. Monge (1746-1818), P.S. Laplace (1749-1827) and A.M. Legendre (1752-1833); and their ranks were augmented considerably in 1787 when J.L. Lagrange (1736-1813) moved to the Paris Academy from his post at Berlin’s equivalent.

The French maintained their dominating status in mathematics during the period treated here, not only by the happenstance of producing major figures but especially for the explicit encouragement given to science and engineering by the new regime that followed the revolution of 1789. The Ecole Polytechnique, founded in 1794, was an important focus, partly for its employment of all the figures named above as teachers or as graduation examiners. This next generation included J.B.J. Delambre (1749-1822), L. Carnot (1753-1823), G. Riche de Prony (1755-1839), J.B.J. Fourier (1768-1830), J.N.P. Hachette (1769-1834), L. Puissant (1769-1843)
and A.M. Ampère (1775-1836); apart from Delambre, all were also involved with the school for some period and purpose.

An especially important reason for the success of the school was its policy of recruiting talented students. Several of them went on to pursue important careers in mathematics, engineering and science, some within the school itself, some at the various other schools and professional institutions (especially in civil and military engineering) available in the country, and some at both. These ranks include J.B. Biot (1774-1862), L. Malus (1775-1812), L. Poinsot (1777-1855), C.L.M.H. Navier, (1785-1836), S.D. Poisson (1781-1840), A.J. Fresnel (1788-1827), J.V. Poncelet (1788-1867), A.L. Cauchy (1789-1857) and G.G. Coriolis (1792-1843).

The task addressed here is to describe the place accorded to the contributions of Euler by this cohort, and also by many contemporaries who made up the community in total. The primary and historical literature on the achievements involved is far too vast to be treated in detail. The most substantial single source is [Grattan-Guinness 1990a], where many historical items are cited and recommended for further information. Historical articles on several of the major books are provided in [Grattan-Guinness 2005, chs. 12-14, 16-21, 24-28]. To my knowledge the question of Euler's influence on the French has been treated explicitly, though briefly, only in [Grattan-Guinness 1983], [Grattan-Guinness 1985a]; but the literature on Euler and/or the early 19th century contains many individual pieces of evidence on his place and influence.

2. Varieties in the calculus and mechanics

For all mathematicians of the mid and late 18th century, by far the main part of mathematics was the calculus and its use in mechanics. Three traditions of the calculus were in place: Newton’s fluxions (though largely confined to British mathematicians); the differential and integral theory, set out by Leibniz and modified by Euler with his addition of the differential coefficient, the forerunner of our derivative; and Lagrange’s version, reducing the calculus to a branch of algebra by assuming that a mathematical function \( f(x + h) \) could always be expanded as a power series in \( h \), with the ‘derived functions’ defined from the coefficients of \( h \). In each tradition the integral was specified (‘defined’ is too strong a term) as some sort of inverse of the fluxion, differential or derivative. A fourth tradition of founding the calculus was to be created by Cauchy, as we see in section 9 below.

As well as the core topics of differentiation and integration, an impressive
body of knowledge lay in the general theory of solutions of ordinary and partial differential equations. In addition, many particular methods were found to solve equations of various kinds. These achievements, especially the second, led to quite a wide range of special functions and infinite series; they also encouraged the theory of polynomial and other equations, in particular properties of their roots.

Mechanics lay alongside this empire, and indeed constituted an even more enormous subject in its own right. It too had three traditions in place by the late 18th century [Grattan-Guinness 1990b]: one based upon central forces and Newton’s various laws; another relying upon ‘live forces’ and their relationship to work; and a third, often called ‘analytical’, where principles such as d’Alembert’s, least action and virtual velocities held sway.

The range of phenomena and artefacts handled within mechanics can be fairly divided into five branches; the descriptive italicised adjectives that follow are mine. Proceeding from the rather large to the very small, we have celestial mechanics, where all heavenly bodies were treated as mass-points; planetary, where the major questions included the shapes of these bodies, now taken to be extended, and related topics such as Lunar theory, topography and the tides; corporeal, including the basic principles of the subject (as required by the three traditions just mentioned), and topics such as sound, elasticity theory and fluid mechanics; engineering, covering, for example, friction studies of various kinds, machines including water-wheels and turbines, and structures such as arches; and a little work on molecular structure.

3. Euler’s place: preliminary remarks

The question addressed here concerns the ways in which Euler’s contributions were to be adopted, adapted or maybe ignored by the cohort of French mathematicians from around 1790 up to 1830. Two parts of that output need to be noted: the very many papers and books that appeared by the time of his death; and the papers, around 100 in number, that the Saint Petersburg Academy published posthumously in their Mémoires until 1830. I shall not take account of his manuscripts or letters, as the former were not then available while the latter were known only to their correspondents and maybe a few others.

Firstly, let us note that all sorts of specific results due to Euler were known, stated in textbooks and monographs. In addition, some of his main results or methods had become part of the mathematical furniture; the calculus with the differential coefficient, for example, and the exposition
of perturbation theory in celestial mechanics by expanding the principal variables in infinite trigonometric series. So the place of Euler was sure, although often not explicitly stated; as time went by, the newer authors may not have known that Euler was their original source for notions that they learnt from textbooks and other writings by intermediate authors. In addition, at that time references to others’ works were not given systematically in science in general, and the tradition of ending a paper or book with a list of works did not commence until the late 19th century. Since lack of evidence is not evidence of lack, the place of Euler is doubtless underestimated in the account to follow.

4. Euler or Lagrange in the calculus and analysis?

These two mathematicians were arguably the two main sources and influences on others, at least into the early 19th century. In analysis and especially the calculus there were specific differences in their approaches. In all aspects of mathematics Lagrange was an algebraist; that is, not just did he use algebra, like everybody else, but he sought to reduce mathematical theories to algebraic principles. His reliance upon Taylor series mentioned above is typical (and important); the claim was that only the normal algebraic operations were needed to develop the calculus. (He allowed for exceptional cases when a function and/or its derivatives took infinite values.) An important theory adjoint to the differential and integral realms was the calculus of variations: Euler had made important innovations, and indeed the name is due to him, but its generality and algebraic formulation owed most to Lagrange. He publicised his approach in the books Théorie des fonctions analytiques (1797), and Leçons sur le calcul des fonctions (1804 and later editions), which were based upon his teaching at the École Polytechnique.

Now the calculus was a much broader subject than (common) algebra, so that the algebraic brief had to be extended. Fulfilling this aim encouraged some followers (more than Lagrange himself) in the development of algebras that were new in the sense that their ‘objects’ were neither numbers nor geometrical magnitudes. One algebra was that of differential operators, based upon forming the operations of differentiation as \( D := d/dx \) and integration \( \int := 1/D \), where ‘1’ is the identity operator. The other was functional equations, such as \( f(x)f(y) = f(x + y) \) (to take a very simple example), where the unknown is the function \( f \).

The measure of support for Lagrange’s approach and these algebras was quite well supported by the French, and it can be seen as an eclipse of Euler.
But Euler’s version of the differential and integral calculus retained its great popularity, especially in applications; it was used in some way in almost all the contexts reviewed below. However, the balance was different in the general theory of ordinary and partial differential equations. There Euler had made important contributions – for example, on singular solutions – but Lagrange had rather taken over with his theory of general solutions of various kinds, which had been further developed by Laplace, Monge and others.

So the inheritance of the calculus for the French was several-sided. Two substantial and contrasting monographs published in the later 1790s exemplify the differences.

The physicist J.A.J. Cousin wrote the first one in two volumes. In his second volume he covered quite a wide range of differential equations, and so gave Euler’s contributions quite a reasonable coverage; but in the first volume he praised the use of limits and judged the differential method to be imperfect; he even interpreted ‘\(\frac{dy}{dx}\)’ as the limiting value of the difference quotient [Cousin 1796, vol. 1, esp. pp. 151-153].

A different balance comes from Lagrange’s successor as professor at the Ecole Polytechnique, namely Lacroix. His mathematical mentor was neither Euler nor Lagrange but the Marquis de Condorcet (1743-1794), not a major mathematician but a significant representative of Enlightenment philosophy. Together they prepared an edition of the Lettres (1787-1789), a few years after Condorcet prepared the eloge of Euler for the old Académie [Condorcet 1786]. Later Lacroix wrote a very praising article on Euler for a general multi-volume biography project [Lacroix 1815].

One of the effects on Lacroix of this kind of philosophy was the desire to present all theories, not just one preferred approach. His major publication was a huge Traité du calcul différentiel et du calcul integral, which appeared in two three-volume editions [Lacroix 1797-1800], [Lacroix 1810-1819]. Interested in the history of mathematics, he endowed his book with a level of scholarship unique for that time: in the table of contents he listed the many original texts that he had consulted, and he finished the book with a combined name and subject index. The latter is particularly useful, as it shows that the entry for Euler is about 50 percent longer than that for Lagrange. The main difference lay in the mass of particular results about series and functions, from which Lacroix reported quite a large selection; otherwise he cited both men for their versions of the calculus and contributions to the general theory of ordinary and partial differential equations.

In the latter context Lacroix added a long and interesting footnote on the notations for the multi-variate calculus. He found unnecessary Euler’s use of brackets to denote partial differential coefficients (such as ‘\(\frac{dz}{dx}\)’), but he was quite critical of the capacity of Lagrange’s primes to distinguish

5. Euler or Lagrange in mechanics?

Lagrange’s intent in mechanics was similar; again he wanted to algebraize it, as the title of his treatise, Méchanique analitique [Lagrange 1788], makes clear. The long-suffering word ‘analytic’ meant ‘algebraic’ here, as he stressed in his oft-quoted remark in the preface that ‘one will not find Figures in this Work:’ algebra was the only guarantee of the generality and rigour with clarity that major branches of mathematics should exhibit. Hence he drew upon principles such as d’Alembert’s, least action and ‘virtual velocities’, which could be formulated algebraically, including using the calculus of variations; from them he obtained the basic equations for dynamics that are now named after him. Other principles, in particular, Newton’s laws and the conservation of energy, came out as theorems. He also included in his book some short historical passages, which became more influential than they deserved.

Again the reaction was mixed. The theory was capable of extension, notably with the theory of ‘Lagrange-Poisson’ brackets to solve canonically the equations of motion, which the aged Laplace and his young disciple Poisson developed between 1808 and 1810. But in general the approach showed its strength best in the systematic exposition and assembly of results already found; it was not normally conducive to the creative side of mechanics. There Newton’s and the energy/work traditions were kinder, especially for their ready appeal to geometry and spatial situations.

The position of Euler was rather peculiar. He made substantial use of Newtonian mechanics: indeed, some of its normal features are actually due to him, such as the notion of the mass-point, and taking the second law exclusively in the form $F = ma$ and applying it in any direction. (For some reason Lagrange attributed this last innovation to the Scot Colin MacLaurin [Lagrange 1788, pt. 2, sec. 1, art. 3]. Euler’s handling of perturbation theory was mentioned in section 3, and he adapted Newton’s laws in continuum mechanics, especially in fluid mechanics and elasticity theory. However, in the 1740s he had also been one of the main advocates of the (new) principle of least action, upon which he failed to draw in these and other later contributions!)
In addition to the Newtonian and analytical traditions in mechanics, a third approach based upon energy and work was put forward as a general one by Carnot in the 1780s; in contrast to Lagrange’s belief that dynamics was reducible to statics, he stressed dynamics and mechanical situations involving impact. The role of Euler here was modest, though he had used ‘quantity of action’ in connection with his advocacy of the principle of least action. Several engineer scientists connected with the Ecole Polytechnique furthered Carnot’s tradition with enthusiasm: Hachette, Navier, Coriolis (who coined ‘work’) and Poncelet stand out.

These men will have gained some insights as students from the engineer de Prony, who was with Lagrange a founder professor of mathematics at the Ecole Polytechnique, and led the teaching of mechanics. Further, while Lagrange taught his calculus for only a few years, de Prony held his chair for 20 years, when he switched to graduation examiner.

De Prony published several volumes of his lecture courses at the school, of which the first, Mécanique philosophique [de Prony 1800], is the best known and most interesting, although incomplete. In this work he made explicit his attachment to Enlightenment philosophy, in particular classifying parts of mechanics in various ways: synoptic tables, and especially the division of almost all of the right hand pages into four columns listing the notations, definitions, theorems and problems. At first glance the book seems to be very Lagrangian: lots of algebra, and no diagrams. But the algebra is not variational but rather trigonometry, especially to express components of notions such as force and moment. In his preface he acknowledged his sources: major writings by Lagrange and Laplace, but ‘The principal works that have furnished me with my material are those of Euler’ [de Prony 1800, vii].

The same features apply also to de Prony’s later textbooks, even the Leçons de mécanique analytique [de Prony 1810-1815] (complete, and admitting quite a few diagrams). However, in his final Part on machines he made little use of Euler’s writings on science and technology, which for some reason were little used by anybody.

A broadly similar impression about Euler comes from the second edition of Bossut’s treatise on hydrodynamics [Bossut 1796]. In his lengthy ‘Preliminary discourse’ he reviewed much of the literature of the 18th century, with a notable emphasis on the concerns of engineers; he even cited one of Euler’s books on the navigation of vessels. As with de Prony, his discussion of the basic equations both of equilibrium and motion of fluids avoided variational techniques.

Other notable texts came from graduates of the school. The first was Poinsot’s book Elémens de statique ([Poinsot 1803, and many later editions]. His opening chapter was an important account of the ‘couple’ (his
word), a major feature of statics of which nobody had previously taken proper account. Then he considered various general conditions for equilibrium that drew upon virtual velocities, and also included a detailed survey of machines, which belonged most closely to the energy/work tradition; and throughout he made much use of diagrams.

In 1811 Poinsot’s non-friend Poisson put out the first edition of his *Traité de mécanique* based upon his teaching at the *École Polytechnique*. One might expect to read a version of Lagrange’s treatise for learners, but this is not so: the calculus of variations was used sparely. Following a long practice among the French, his basic principle was d’Alembert’s, which served not only as a fundament for analytic mechanics but also as the justification for Newton’s laws; Poisson used especially the second law fairly often (but never mentioned Newton once). At least that theory was presented; Poinsot’s recent book on mechanics was ignored completely.

6. On Laplace and his own place

The mixed picture is evident also with our third major figure: Laplace [Gillispie 1998]. His reputation, already high, rose still spectacularly when he began to publish his *Traité de mécanique céleste*, of which the first four volumes appeared as [Laplace 1799-1805]. This work was authoritative for all aspects of celestial and planetary mechanics, and also for many aspects of the calculus, and some series and solutions of various partial differential equations.

Laplace is credited with the remark: ‘Read Euler, he is the master of us all’.¹ Both Euler and Lagrange featured strongly in his work (though he did not much fancy the new algebras). For example, he made much use of Euler’s use of trigonometric series; but he also adapted Lagrange’s marvellous attempt to prove mathematically that the planetary system was stable, a decisive rejection of Euler’s (and also Newton’s) view that God was responsible for stability. Among other sources, for Lunar theory he made most use of d’Alembert’s formulation. Many of the analyses were as much his own as anybody else’s. For example, for the theory of equipotential surfaces and the attraction of a heavenly body to an external point he solved the partial differential equation now named after him and solved it with the help of functions that also took his name in the 19th century but

¹ Unfortunately our closest source for Laplace’s remark is [Libri 1846]. This article, part of a review of an edition of 18th-century mathematical correspondence, is cited in the biographical article on Euler [Anonymous 1857], which therefore may also be by Libri.
then became called ‘Legendre functions’; the use of both names reflects a pretty competitive situation since the 1770s.

A striking feature of Laplace’s opening sections was his use of one of Euler’s posthumous papers: [Euler 1793], in which Euler had proved that torque obeyed the same kind of linearity that obtained with moments. Laplace quickly used it to define the invariable plane of a system of point masses (the planetary system being the case most in mind) in terms of maximal torque [Laplace 1799-1805, Book 1, art. 20]. This use of Euler is not only noteworthy in its own right; it is also a very rare case of anybody using a posthumous paper by Euler.

7. Laplace’s programme of molecular physics, and the alternatives

In the last volume of his treatise Laplace began to make public his growing interest in physics, which did not have a high status in science at that time. He analysed the path of light through the atmosphere, and then added two lengthy supplements to the fourth volume on capillarity. Common to both analyses was a principle that “all” phenomena, mechanical or physical, were to be interpreted as actions between the elementary ‘molecules’ of which the pertaining bodies were presumed to be composed. He based upon it an ambitious programme for physics, for which he recruited several able younger colleagues, mostly graduates of the Ecole Polytechnique.

In particular, Laplace adopted a corpuscular theory of light, which was then the more popular type of theory among French physicists (see, for example [Haiüy 1806, vol. 2, 134-401]; it was to be the most successful part of his programme. The most important theorist and experimentalist was Malus; among other achievements, he saw that the principle of least action could be used to explain double refraction (an insight that Laplace was to purloin and extend), and he coined the word ‘polarisation’ because he assumed that the moving particles of light oscillated about an internal axis, like the poles of a magnet.

This kind of theory was bad news for fans of Euler, who had adopted a wave theory of light. He had studied especially reflection, refraction and aberration, the latter leading to an interesting exchange about achromatic lenses with the Englishman John Dollond, who upheld Newton’s corpuscular theory [Speiser 1962] – the only detail in which Euler was mentioned in Haiüy’s long account of optics just cited. But bad news turned to good, in that from the mid 1810s onwards Fresnel began to elaborate such a theory. He construed light to be the result of disturbance from equilibrium of the
tiny particles in the assumedly punctiform aether. He appealed to analogies with mechanics whenever useful: principles such as the cosine law of decomposition, and ‘energy’ conservation for double refraction. However, in his papers and letters he referred only once to Euler’s theory, and then in passing [Fresnel 1822, art. 1].

By the 1820s even Laplace was admitting the quality of Fresnel’s theory; it was the main confrontation of his programme. However, it was not the first, which had occurred over heat diffusion. From the mid 1800s Fourier had much exercised himself over this topic, giving it the first extensive mathematical treatment. Philosophically he adhered to a kind of positivism (the word that Auguste Comte was to coin, with Fourier much in mind): heat was heat, to be exchanged with its opposite, cold. He had derived the diffusion equation by the normal (Eulerian) version of the differential and integral calculus, and took no interest in the molecularist re-derivation that Laplace offered in 1809: the loyal Poisson was to pursue the idea from the mid 1810s onwards, but nobody took much interest in it.

So Euler was present in Fourier’s derivation of the equations, but not in the preferred solutions: trigonometric series (not to be confused with Euler’s technique in celestial mechanics) for finite bodies, and integrals for infinite ones. Now Euler had found the Fourier series in [Euler 1798] as a mathematical exercise (as Lacroix was to point out to Fourier), but he did not exploit it physically; in particular, he never changed his stance of the 1740s when in the famous debate over the analysis of the vibrating string he had preferred the functional solution of the wave equation, a kind of solution that was normally favoured at that time.

The other main developments in the new mathematical physics lay in the study of ‘electricity’ (mostly electrostatics) and magnetism. Here the Laplacians enjoyed some success, thanks mainly to the efforts of Poisson (around 1812 and 1824 respectively). However, molecularism took a limited role; Poisson made much use of the electric and magnetic fluids that were supposed to exist. Euler had said little technical about either subject, though several of the Lettres treated magnetism. Further, naturally he did not anticipate electromagnetism, which was to obsess Ampère from 1820 to around 1827 – and to attract little interest among the Laplacians. However, once again his form of the calculus was preferred there, and indeed was extended importantly into line and surface integrals; Poisson had already made some use of the latter in his contributions to magnetism.
8. Continuum mechanics, molecular and otherwise

Contemporary with these innovations in mathematical physics, mechanics continued to be studied in all its branches. The most important parts not yet treated were fluid mechanics and elasticity theory, where the Parisian talent for rivalry was well to the fore, especially between Poisson, Cauchy and Navier, with Fourier sniping on occasion.

On fluid mechanics, in the 1750s Euler had applied Newton’s second law to a differential parallelepiped and using also his own notion of pressure; Lagrange had later substituted the method that came to be known as ‘the history of the particle’, which made use of the calculus of variations. The results of both men were restricted to shallow fluid bodies. Shortly after the death of Lagrange in 1813, the mathematical and physical class of the Institut posed a prize problem for 1814 on the propagation of waves in a deep fluid body. Cauchy won it; Poisson, already a member of the class, contributed two papers at the same time. He based his treatment upon Euler’s method, while Cauchy drew upon Lagrange’s; one might have expected the preferences to be the other way round. As usual, Cauchy produced the more profound results (in particular, he found Fourier’s integral theorem, in apparent independence of Fourier), but not especially because of his use of Lagrange’s method.

In elasticity theory, the class had already run in 1811 a problem on the motion of an elastic lamina. This problem, partly inspired by the sand experiments of the Austrian acoustician Ernst Chladni, seems to have been tailored for Poisson, who was then not yet a member of the class, to produce another Laplacian molecular exercise. He did produce one eventually, but the prize was won by Sophie Germain, after three versions and important help from Lagrange and Legendre. The basic ideas, however, were hers, and drew upon Euler’s work.

More significant developments began in the late 1810s, with a string of papers from Navier, Cauchy and Poisson. Navier worked his way from elastic rods and planes to solids and also viscous fluids. Some of Euler’s assumptions were used in the formation of the equations, but for solution he appealed to Fourier’s new methods. Poisson predictably was very molecularist. Cauchy as usual eclipsed everybody, with a long string of analyses in terms of stress and strain (to use the names which William Rankine was to introduce). Some of the models were molecular while others not, and it is not easy to tell why each type was chosen. He then adapted his method to study dispersion within the framework of Fresnel’s optics. As usual he was spare in references, and he may not have drawn much upon either Euler or Lagrange.
9. A new tradition for the calculus: the impact of Cauchy

Student at the *Ecole Polytechnique* in the mid 1800s, a decade later Cauchy was appointed professor of analysis and mechanics there in the changes that accompanied the restoration of the Catholic monarchy, to which he was fanatically attached. His teaching was disliked by students and staff for its inappropriate content for an engineering school, and also for his failure to coordinate with other courses; but mathematically it was of immense importance.

Cauchy formulated a fourth version of the calculus. It was grounded upon a proper *theory* of limits that itself was based upon the careful studies of infinite sequences of values and not just the modestly developed *notions* that his predecessors had achieved. In its terms he defined the derivative as the limit of the difference quotient and the integral as the limit of a sequence of partition sums, and he allowed in both cases for the possibility that the limit did not in fact exist. As one offshoot, by means of counter-examples he refuted in 1822 Lagrange’s belief in the universality of the Taylor expansion. His new version (which was not motivated by these counter-examples) was bad news for all predecessors, Euler included; but it gradually became adopted worldwide, especially among those mathematicians who stressed rigour. However, Euler’s version long continued to retain its high status among those figures concerned with applications, who included Cauchy’s colleagues at the school.

As part of his reliance upon a theory of limits, Cauchy also revised the theory of functions and of infinite series, defining continuity of the former and convergence of the latter in terms of the proven existence of the limiting value. All previous criteria were substantially revised; in particular, much of Euler’s production of sums of series was rejected as illegitimate, and only from the end of the 19th century was it rehabilitated within the theory of summability and formal power series.

10. Three smaller topics

10.1. *Geometry*

Euler’s *Introductio in analysin infinitorum* (1748) was divided into two distinct volumes. The first one covered many aspects of (real- and some complex-variable) analysis and the theory of functions, and became a standard reference for these topics. The second one helped substantially to launch analytic and coordinate planar and solid geometry [Boyer 1956, chs.
7-8]. Among French mathematicians Monge and Puissant were active, and [Lacroix 1799b] and [Biot 1802] produced textbooks that appeared in many later editions. The subject also featured in the treatises on the calculus by Cousin and Lacroix mentioned in section 4. Once again Lagrange tried to algebraise the theory, but this time with limited success.

In addition, the *Introductio* itself was translated into French and published as [Euler 1796-1797]. The task was fulfilled by the school-teacher J.B. Labey, who later also published a translation of the *Lettres*.

10.2. **Number theory**

Number theory was a very recondite subject, with few practitioners; however, three of them were Euler, then Lagrange, then Legendre. In his *Essai sur la théorie des nombres* [Legendre 1798, and later editions] Legendre treated the algebraic side of the subject. The topics covered included reduction of quadratic forms, sums of squares, cyclotomy, reciprocity properties, certain equations and their roots, and Fermat’s ‘last’ and other theorems. In his preface he duly praised Euler, and acknowledged Lagrange (and also C.F. Gauss in the later editions). This was valuable tribute from the community of French mathematicians; but it was a small one, since Legendre was its only regular practitioner to the subject in the period treated here.

10.3. **Probability and mathematical statistics**

Some of Euler’s contributions to analysis bore upon these topics: in particular, the beta and gamma functions and the hypergeometric series. In addition, he wrote on the errors of observation, games, tontines and lotteries, and mortality and annuity tables [Sheynin 1972]. Some of this work lay in interactions with contemporaries, especially Bernoulli and Lagrange. However, few French worked in these fields; mainly Laplace and some Poisson, with a short burst from Fourier and a textbook by Lacroix. They do not appear to have made much use of Euler’s offerings, which seem never to have been much used.

11. **Three general surveys**

I complete this appraisal with three French sources of the 1800s that give us further insight about Euler’s status. The first is the massive *Histoire des mathématiques* of E. Montucla. He died in 1799, just as he was writing and proof-reading a Part of the third volume. The project was taken over
by Montucla’s friend the astronomer J.J. Lalande, who edited the surviving manuscripts and wrote the rest himself, drawing on colleagues such as Lacroix for certain sections. The third and a fourth volume appeared, as [Montucla 1802]. The books contain a huge amount of information, although often surprisingly spare of symbols.

A noteworthy feature of the volumes is the entry for Euler in their index: ‘Euler, the greatest geometer of the eighteenth century’ [Montucla 1802, vol. 4, 678]; no other figure was characterised in such a way, though the names of persons in the index are poorly furnished. Further, while Euler was mentioned a lot, some details are missing. Take, for example, the Part in progress when Montucla died, a long and rather untidy account of the calculus and analysis during the 18th century; he noted Lagrange’s approach to the calculus [Montucla 1802, vol. 3, 260-270], but he did not describe Euler’s introduction of the differential coefficient. Again, in the next Part, on optics, his wave theory was mentioned less than one might have expected.

In the last two Parts, on mechanics and machines, the text adopted Lagrange’s *Méchanique analytique* as the main guide, including its little historical essays. So, while Euler duly appeared in the discussions of some of the basic principles, his later contributions were rather summarily treated. Further, as usual he did not feature in (Lalande’s) review of machines and technology, even though the bibliographical information there was quite extensive. Carnot was also omitted; de Prony’s engineering treatise *Nouvelle architecture hydraulique* [de Prony 1790-1796] was a leading source.

Also published in 1802 was our second source, the second volume of a much shorter (but also prosodic) history of mathematics written by Bossut. Most of the book treated the history of the calculus from its creation by Newton and Leibniz, and much of the text was taken up with applications [Bossut 1802]. Euler featured a fair amount in the parts of the book covering his career period, though perhaps less than one might have expected. But he was better served than was Lagrange, since Bossut adopted the peculiar policy of omitting all figures then still living. Perhaps in response to the criticisms, a few years later he issued an expanded version of his book, coming right up to date. Euler featured rather more than before, especially in applications (including optics and some technology); he had more page entries than anybody else in the index, and was praised on several occasions [Bossut 1810, esp. pp. 148-150]. But he still left out the differential coefficient.

Our last source also comes from that time. As a permanent secretary of the scientific class of the *Institut de France*, in 1809 Delambre had to present to Emperor Napoléon a book-length survey of progress in the ‘mathematical sciences’ (pure and applied) during the past glorious 20
years. As for Lalande, Lacroix helped him with the purer mathematical sections [Delambre 1810]. Lagrange and Laplace were naturally the leading authors, but Euler was next, with more entries than even for Legendre; he was mentioned over a score of times, of which several were more than passing references. However, the balance again was rather askew: reasonable for the calculus and mechanics, but nothing on technology, or on cartography.

12. Concluding remark

As one might expect, Euler was a major background figure for the French in the period treated here, and for several topics he was a good deal more prominent. The pure mathematics seems to have been the most visible part of his achievement, and several parts of his work in celestial, planetary and continuum mechanics; the technology survived much less well.

The main figure ‘in between’ Euler and his French successors is Lagrange, their senior member from 1787 until his death in 1813. More importantly, he differed from Euler substantially on the adopted principles of both the calculus and mechanics. The ‘competition’ between them is hard to evaluate. Lagrange had put forward impressive general theories, but their utility was limited, especially in applications or the creative sides of theories. But the use of Newton’s laws in mechanics can only be seen as a partial affirmation of Euler’s position, where nominally the principle of least action should have been as prominent as it was with Lagrange. Finally, while Lagrange’s published references to Euler were rather slender, on his deathbed he praised Euler to the skies: ‘read Euler, because in his writings all is clear, well calculated, because they teem with beautiful examples, and because one must always study the sources’ [Grattan-Guinness 1985b, art. 4].

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Euler’s Influence on the Birth of Vector Mechanics

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1. Introduction

In the course of his examination of Euler’s paper “A more accurate development of the formulae found for the equilibrium and motion of flexible threads” [E608], C. Truesdell wrote:

“The reader will have remarked Euler’s mastery of the methods of vectorial algebra; the formulae we have presented are shortened by the use of vector symbols, but the operations indicated are those used by Euler.” [Truesdell 1960, p. 383]

There is, of course, some exaggeration in this statement. If we agree that vector calculus is a theory of the composition of directed segments expressed by means of an algebraic symbolism, it is a matter of fact that there is no trace of such ideas in the writings of Euler. On the contrary, as we will see, there are many reasons to believe that Euler did not fully understand the vectorial character of the entities and the operations that occur in his purely algebraic calculations.¹

¹ In this article I use the term “vector” quite freely. It would have been more appropriate to employ everywhere the locution “directed segment,” for this is what these early authors had in their minds, but its usage would have led to a cumbersome mode of expression. The difference is significant: a vector is, strictly speaking, an element of a vector space.
Truesdell had previously given a different and more just estimation of the role of Euler in the development of vector calculus:

“The expression of the laws of motion in \textit{rectangular Cartesian co-ordinates} is also of the greatest importance. Today this possibility is so obvious that many scientists seem to believe that Newton himself used Cartesian co-ordinates, but of course this is not so. [...] The importance of the use of Cartesian co-ordinates lies deeper than in mere simplicity; in these co-ordinates the addition of vectors located at different points is so natural as to become customary at once, and the possibility of performing this addition lies at the heart of the classical conception of space-time.” [Truesdell 1960, p. 252]

In fact, Euler has the merit of having constantly referred all quantities to rectangular axes fixed in space in his works from about 1750 onward. It is clear that from the Cartesian representation of physical quantities their vectorial character can easily be judged; let us recall Heaviside’s observation:

“I ought to also add that the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics, but to find out quaternions required a genius.” [Heaviside 1892, vol. 2, p. 557]

However, the task of extracting the concept of vector from analytic geometry and mechanics turned out to be more difficult than Heaviside had imagined.

2. On Euler’s conception of vectors

What then was Euler’s conception of the geometrical representation of vectors? Perhaps, a clearer account can be found at the beginning of his “Attempt at a metaphysical demonstration of the general principle of equilibrium” [E200], where he defines the concept of force:

“One calls force everything that can change the state of bodies, both of their movement and of their rest. [...] In each force there are two things to consider: the quantity and the direction. By quantity one understands how much a force is greater or smaller than another, and the direction allows us to know in which sense every force acts on bodies to disturb their state.” \footnote{“On nomme force, tout ce qui est capable de changer l’état de des corps, tant de leur mouvement que du repos [...] Dans chaque force il y a deux choses à considérer, la quantité & la direction: par la quantité on comprend combien une force est plus grande} [E200, p. 246 (author’s translation)].
This description is not much different from those employed today in high school textbooks. Yet it would be wrong to assume that Euler was able to interpret all quantities that appeared in his formulae in terms of the composition of directed segments.

An example of Euler’s inability to judge the vectorial character of a geometric entity occurs in his memoir “On the movement of rotation of solid bodies around a variable axis” [E292, §XXVIII]. Having just discovered the equations for the motion of a rigid body in the form

\[
\begin{align*}
d(\omega \cos \alpha) + \frac{c^2 - b^2}{a^2} \omega^2 dt \cos \beta \cos \gamma &= \frac{2gP dt}{Ma^2}, \\
d(\omega \cos \beta) + \frac{a^2 - c^2}{b^2} \omega^2 dt \cos \alpha \cos \gamma &= \frac{2gQ dt}{Mb^2}, \\
d(\omega \cos \gamma) + \frac{b^2 - a^2}{c^2} \omega^2 dt \cos \alpha \cos \beta &= \frac{2gR dt}{Mc^2},
\end{align*}
\]

where the coordinate axes are laid along the principal axes of inertia relative to the centre of mass, \( \omega \) is the angular velocity, \( P, Q, R \) are the moments of the applied forces about the coordinate axes, \( \cos \alpha, \cos \beta, \cos \gamma \) are the direction cosines of the instantaneous axis of rotation, \( Ma^2, Mb^2, Mc^2 \) are the principal moments of inertia and \( g \) is a constant, he simplified them (“pour abrégé nos formules”) by placing

\[
\omega \cos \alpha = x, \; \omega \cos \beta = y, \; \omega \cos \gamma = z.
\]

The three quantities \( x, y, z \) are clearly the projections of a directed segment on the coordinates axes. Astonishingly, Euler seems not to recognize the geometrical meaning of this passage, thus missing the discovery of the angular velocity vector.

Among the formulae used by Euler are the expressions for the velocity of a point of the body in terms of its coordinates and the angular velocity,

\[
\begin{align*}
u dt &= \omega dt (z \cos \beta - y \cos \gamma), \\
v dt &= \omega dt (x \cos \gamma - z \cos \alpha), \\
w dt &= \omega dt (y \cos \alpha - x \cos \beta),
\end{align*}
\]

or plus petite qu’une autre, & la direction nous donne à connaître en quel sens chaque force agit sur les corps pour en troubler l’état.”

3 In passing, let us note that up to about 1840 forces were graphically represented by line segments. The representation by means of arrows appears, perhaps for the first time, in the works of Matthew O’Brien [O’Brien 1851a, O’Brien 1851b].

4 Euler repeated this derivation in his treatise on the motion of rigid bodies, the *Theoria motus corporum solidorum* . . . [E289, cap. XV, §808].
where $u, v, w$ are the components of the velocity ($§$XII). Had Euler pursued
the question further, he might have discovered the geometrical relations
that are now expressed by the vectorial formula $v = \omega \times r$. (As a matter of
fact, this step was taken in [Cauchy 1844].) Thus we see that at this time
Euler was not aware of the geometrical meaning of the formulae equivalent
to vector products.

The gap that separated the eighteenth century from the general concept
of vector can also be seen in the work of Lagrange. Two instances stand
out in this regard.

The first instance occurs in his famous paper on the analytic theory of the
triangular pyramid [Lagrange 1775b]. Here Lagrange gave the geometrical
meaning of the expressions for the scalar and the mixed product ($§$11;
§15), but missed seeing that an ordered triple of the form $(yw - zv, zu -
wx, xv - zu)$ represents a vector. It is difficult to understand how he could
interpret the very complicated formulae that appear at the beginning of
his work without some knowledge of the external product ($§1$-3).

The second example is taken from the *Mécanique analytique*. In the
first edition Lagrange considered the kinematics of a rigid body with a fixed
point [Lagrange 1788, p. I, sect. IV, art. 9]. In so doing, he resolved a general
infinitesimal rotation into three rotations about the axes of a rectangular
system of coordinates, thus demonstrating their law of composition. From
his formulae it is easy for us to see the vectorial character of infinitesimal
rotations, yet he failed to do so. However, in 1811, after a lapse of more
than twenty years, Lagrange returned to the subject in the second edition
of his treatise, now entitled *Mécanique analytique*. This time he added to
his preceding analysis a comment in which the possibility of representing
infinitely small rotations by means of a directed segment is emphasized:

“It is clear from this development that the composition and resolution
of rotational motions are entirely analogous to rectilinear motions.

“Indeed, if on the three axes of the rotations $d\psi, d\omega, d\phi$, one takes
from their point of intersection lines proportional respectively to $d\psi, d\omega,$
$d\phi$, and if one draws on these lines a rectangular parallelepiped, it is easy
to see that the diagonal of this parallelepiped will be the axis of composed
rotation $d\theta$ and will be at the same time proportional to this rotation $d\theta$.
From this result, and because the rotations about the same axis can
be added or subtracted depending on whether they are in the same or

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5 See also [Lagrange 1775a, n. 5]

6 A geometrical interpretation of these formulae were given much later in [Binet 1813].

7 It is possible that this result was taken over by Lagrange from the work of
Paolo Frisi, who discovered it as early as 1759. This is a matter of debate, for La-
grange did not cite Frisi in this context. Frisi gave several accounts of his theorem
[Frisi 1759,Frisi 1767,Frisi 1768,Frisi 1783a,Frisi 1783b].
opposite directions, in general one must conclude that the composition and resolution of rotational motions is done in the same manner and by the same laws that the composition or resolution of rectilinear motions, by substituting for rotational motions rectilinear motions along the direction of the axes of rotation.” [Lagrange 1811-15, part I, sect. III, §III, art. 15; Oeuvres, t. XI, p. 61; translation by A. Boissonnade and V. N. Vagliente, 1997].

It is likely that this new interpretation of the old formulae had been prompted by the appearance of Poinsot’s Statique in 1803.\(^8\)

Lest the foregoing criticism seem too harsh, we must remember that before the nineteenth century even the simplest forms of vector calculus were completely unknown. More importantly, to be accepted as a vector a geometric entity had to obey to the parallelogram law, and thus it was not sufficient that it had three “components”.

Our account of vectors in the eighteenth century should make it clear how far removed mathematics was at that time from a real comprehension of the subject. All this began to change by the end of the century. As far as I know, the first recognition of the geometrical meaning of the vector product occurs in Euler’s paper “An easy method for investigating every property of curved lines not lying in a plane” [E602]. Here Euler explicitly stated that the three expressions

\[
\frac{dz}{ds} \frac{d^2y}{ds^3} - \frac{dy}{ds} \frac{d^2z}{ds^3}, \quad \frac{dx}{ds} \frac{d^2z}{ds^3} - \frac{dz}{ds} \frac{d^2x}{ds^3}, \quad \frac{dy}{ds} \frac{d^2x}{ds^3} - \frac{dx}{ds} \frac{d^2y}{ds^3},
\]

which, from a modern point of view, are the components of the binormal vector with respect to three orthogonal axes, define a unit segment perpendicular to the osculating plane of a skew curve (§27).\(^9\) However, the same paper contains analytical expressions roughly equivalent to the formulae

\[
b = t \times n, \quad n = b \times t, \quad t = n \times b,
\]

where \(t, n, b\) are respectively the tangent vector, the principal normal and the binormal (§26), but Euler does not mention this interpretation. Thus there are reasons to believe that he did not fully understand the matter. Euler could scarcely have failed to notice that the three expressions reported above are similar in form to the projections on the coordinate planes of the areas described by the radius vector in the “law of areas” (that is, the conservation of moment of momentum for an isolated system), but surprisingly he did not call attention to this fact. In passing, we note the general formulation of the theory of skew curves by

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\(^8\) We know from [Bertrand 1872] that Lagrange had discussed with Poinsot the new developments in vector mechanics around 1806.

\(^9\) These expression can also be found in §19 of Euler’s memoir referred to by Truesdell in the first citation [E608].
means of directed segments occurs much later, in [Saint-Venant 1845] and [Chelini 1845].  

In order to understand the development of vector calculus it is necessary to take these results into account, for some of the discoveries of the period 1760-1820 led directly to the development of the earliest theories of vectors. The starting point of this new stream of thought can be found in two papers by Euler on the theory of moments. They were presented to the Academy of Science of St. Petersburg in 1780, and appeared consecutively in the 1789 volume of the *Nova acta academiae scientiarum imperialis Petropolitanae*, which was published only in 1793, ten years after Euler's death [E658,E659]. This delay in publication turned out to have some consequences for the subsequent development of the theory.

3. Euler’s first memoir: the solution by pure geometry

The purpose of Euler’s “On finding the moments of forces about any axis; where several important properties of couples of straight lines, not lying in the same plane, are explained” [E658] is clearly set forth in the title. The paper opens with a purely geometrical definition of the moment of a force $V$ about an axis $az$: Take any point $P$ on the line of action of the force, and multiply the component of $V$ perpendicular to the plane $aPz$ by the distance of $P$ to $az$. Euler remarks that it appears very difficult to give a general analytical expression for this definition. However, taking into account the arbitrariness of the choice of the point $P$, it is possible to make the segment from $P$ to $az$ equal to the common perpendicular of the two assigned lines. Thus we reach the main problem: To find the distance between two assigned straight lines in space, supposing that one of them passes through the origin. From this point onwards, much of this work, more than half of the whole, concerns geometrical questions, and Euler carefully separates the basic geometrical results from their applications to mechanics. It must be noted, though, that even these geometrical parts remain algebraically oriented, for Euler describes the positions of the points and the straight lines by means of a rectangular Cartesian system of coordinates.

Mention must be made of the fact that in this paper a straight line in three dimensions is assigned by means of one of its points and its directon

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10 Saint-Venant coined the term “binormal” in [Saint-Venant 1845, p. 17].

11 The origin of vector calculus in geometry and mechanics is usually not recognized in the standard histories of mathematics. Some idea of the question can be gained from [Caparrini 2003,Caparrini 2004]. *Hactenus hec. Cetera in tempus aliud reservo.*
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This is virtually the modern form of expression, completely symmetrical in all the variables and ready to be translated into the language of vectors. We must recall that twenty years later, in the second edition of his Feuilles d'analyse [Monge 1801], Monge still described a straight line by its projections on two coordinate planes.

Before turning to the distance problem, Euler finds an expression for the angle \( \omega \) between two straight lines in the form

\[
\cos \omega = fF + gG + hH,
\]

where \( f, g, h \) and \( F, G, H \) are the cosines of the angles formed by the two lines with the coordinate axes (§13), which is clearly equivalent to the modern scalar product. However, this result was not new, for it can be found in Lagrange's famous paper on the analytic study of tetrahedra [Lagrange 1775b, n. 11], where it is also interpreted geometrically. Let us note that in his proof – not much different from Lagrange's – Euler starts from the formula

\[
\cos \omega = \frac{AZ^2 + Az^2 - Zz^2}{2AZ \cdot Az}
\]

where \( AZ \) and \( Az \) are two straight lines passing through the same point \( A \), which is the analytical expression for the so-called “Carnot theorem”.

The difference between the mastery of analytic geometry in 1780 and today (or 1820, let's say) is clearly seen by looking at Euler's treatment of the distance problem, for his calculations are somewhat prolix by modern standards. In essence, Euler determines the positions of the end points of the segment of minimal distance between the two straight lines (§15-18), then calculates the length \( m \) of the segment (§19-23). Euler's final result, in his own notation, is

\[
m \sin \omega = (Gh - Hg)a + (Hf - Fh)b + (Fg - Gf)c,
\]

where \( a, b, c \) are the coordinates of a point on the axis of the moment. The formula for the distance between two straight lines is a major result in analytic geometry, but is not cited in the histories of the subject.

With this main geometrical theorem stated, in the last part of the paper Euler returns to the problem of finding an analytical expression for the moment of a force. Multiplication of both sides by the intensity \( V \) of the force leads immediately to the desired result,

\[
(\text{moment of } V \text{ about } az) = V(Gh - Hg)a + V(Hf - Fh)b + V(Fg - Gf)c.
\]

\(^{12}\)The formula for the distance of two points in three dimensions makes an early appearance here; see [Boyer 1956, p. 169].

\(^{13}\)Euler's distance formula is clearly a mixed product of vectors, or a third order determinant. As we have seen, the geometrical interpretation of expressions of this kind had been given by Lagrange a few years before. See also [E268, p. 3]
Considering now the special case in which the axis \( az \) is successively parallel to each of the coordinate axes, the above formula gives

\[ V_f(bH - cG), V_g(cF - aH), V_h(aG - bF), \]

and hence the general formula becomes

\[ fP + gQ + hR, \]

where \( P, Q, R \) are respectively the moments about the axes \( Ox, Oy, Oz \).

This expression indicates that moments of forces can be resolved into components along three orthogonal axes by the parallelogram law. In fact, it is equivalent to a scalar product which expresses the projection of a vector along a given straight line by means of components of the vector on three orthogonal axes and the direction cosines of the line.

Euler saw its meaning, for the paper ends with these words:

"Therefore the moments about three orthogonal axes can be composed exactly as the simple forces. For if three forces \( P, Q, R \) were applied to the point \( a \), acting along the directions \( af, ag, ah \), they would form a force equal to \( fP + gQ + hR \) acting along the direction \( az \). This marvellous harmony deserves to be considered with the greatest attention, for in general mechanics it can deliver no small development." 14 [E658, §35 (author’s translation)]

This passage makes it plain that Euler now visualizes the moment of a force about an axis as a vector lying along the axis. The last remark, of course, is prophetic.

The discovery of the vectorial properties of moments is a result as fine and important as any Euler ever achieved. The final expression can justly be called Euler’s formula for moments.

Having followed Euler’s derivations of the formula of moments, the reader will be no doubt surprised to learn that Euler had already obtained this result almost twenty years before, as the solution of Problem 2 of his paper “On the equilibrium and motion of bodies connected by flexible joints,” [E374] written in 1763 but published in 1769. While the result was the same, the proof was more primitive, and Euler failed to grasp its significance. 15

14 “Momenta igitur virium pro ternis axibus inter se normalibus codem prorsus modo componi possunt, quo vires simplices componi solent. Si enim puncto a applicatae fuerint vires \( P, Q, R \), secundum directiones \( af, ag, ah \), ex iis componitur vis secundum directionem \( az = fP + gQ + hR \), quae egregia harmonia maxima attentione digna est censenda, atque in universam Mechanicam hinc non contemnenda incrementa redundare possunt.”

15 The proof is based on the consideration of an ad hoc system of forces, which is supposed to be equivalent to the assigned forces. Truesdell, who first noticed this formula, remarked that “the solution of Problem 2 is a proof of the vectorial character of moments, in three dimensions” [Truesdell 1960, p. 342]. This is true, but Euler was not conscious of the fact.
It is curious that Euler returned to the same problem without citing his previous derivation, yet this case is by no means unique.\textsuperscript{16} Evidently, by 1780 he had forgotten what he himself had achieved in 1763.

4. Euler’s second memoir: the solution by the first principles of statics

Euler considered his result so important that he derived it anew in a second memoir. Shortly after completing the first paper, he wrote “An easy method for determining the moments of every force about any axis” [E659], in which he presents a new approach to the same problem. “While this important result [i.e., Euler’s formula $fP + gQ + hR$] has been derived by means of geometrical considerations and with quite long calculations, there is no doubt that it can also be deduced directly from the principles of statics. Having thus diligently considered the question, I happened to find quite an easy way, which led me to this result.”\textsuperscript{17} His new proof, essentially, rests upon the resolution of a force by means of the parallelogram law and the possibility of translating a force along its line of action without affecting its moment. Thus his methods here bear a strong resemblance to the purely geometrical formulation of Poinsot.

Supposing that the new axis $I$ passes through the origin of the coordinates, Euler begins by replacing the given force with an equivalent system formed by three other forces, each lying in one of the coordinate planes and parallel to one of the coordinate axes. Hence, each of them has a non-zero moment only about one of the axes. The new forces are then resolved into two components, parallel and perpendicular to $I$, and the moments about $I$ of the perpendicular components are easily found. Expressing these three moments by means of the original components, Euler obtains $fP$, $gQ$, $hR$, and their sum yields the formula of moments.

In the remainder of the paper Euler derives afresh the expression for the distance of two straight lines, starting from the formula of moments. Thus the second memoir exhibits the same results of the first one, but in the reverse order.

\textsuperscript{16}[Truesdell 1960] gives several examples of similar episodes.
\textsuperscript{17}“Quae egregia veritas cum ex consideratione geometrica per calculos satis prolixos derivata sit, nullum est dubium, quin etiam via directa ex principiis staticis deduci quest. Postquam igitur hoc argumentum sollecite essem perscutatum, incidi in viam satis planam, quae me ad hanc veritatem perduxit.” (Author’s translation.)
We pause for a moment to note that when Euler gave these complicated geometrical proofs, somewhat difficult to follow even for the experienced reader, he had been blind for about twenty years.

Euler never developed further his discovery of the vectorial representation of moments, nor put it to any use. This idea was to mature many years later.

5. Impact and influence of the work

It is instructive to follow the history of Euler’s formula up until the beginning of the nineteenth century, for it influenced in various ways several important mathematicians. This was in fact the first step towards a formulation of mechanics entirely based on the concept of vector. Euler’s formula is like an Ariadnean thread through the early development of vector calculus.

According to [Poisson 1827, p. 357], by the time Euler’s two papers were published, the situation caused by the revolution made it difficult for French mathematicians to have access to them. Not knowing Euler’s work, in 1798 Laplace considered the problem of simplifying the equations of motion of an isolated mechanical system by choosing coordinate axes which reduce to zero some constants of motion [Laplace 1798]; thus he discovered the invariable plane. In modern terms, the invariable plane is simply a plane orthogonal to the total moment of momentum vector. To obtain this result, Laplace had to calculate the formulae for the transformation of the projections on the coordinate planes of the areas swept over by the radius vector in the movement of the planets in passing from one coordinate system to another, and thus nearly discovered the vectorial nature of moment of momentum.

Shortly thereafter Laplace wrote a second paper on the same subject, whose title was simply “Sur la Mécanique” [Laplace 1799a]. It is only two pages long and there is not a single formula. Here Laplace remarks that the invariable plane is orthogonal to the axis of moments, which he calls axe du plus grand moment.

The connection between the formulations of Euler and Laplace, which now seems obvious, was established by Prony with a few lines of simple calculations in his Mécanique philosophique [de Prony 1800, p. 110]. There, in a footnote, he gives the first explicit citation of Euler’s first paper. He

\textsuperscript{18}This result was immediately included in the Traité de mécanique céleste [Laplace 1799b, liv. I, ch. IV, n. 21]. See also the Exposition du système du monde [Laplace 1835, VI:199].
remarks that Euler’s formula “is of such simplicity and elegance that it can be considered one of the most beautiful results in mechanics.” 19 While Prony did not add anything new to the preceding works, he has the merit of having clarified and made generally known the first results in the vectorial theory of moments.

Poinsot, independently of Euler and Laplace, initiated a purely geometric approach to the vectorial theory of moments in his famous textbook of statics Eléments de Statique, first published in 1803 but reprinted at least twelve times before the end of the century [Poinsot 1803, n. 60-67]. To study the equilibrium of a rigid body with respect to rotations, Poinsot introduced the couple of forces. A couple is a system of two equal, parallel and oppositely directed forces, whose magnitude is measured by the product of the intensity of the forces by the distance between their lines of action. Poinsot showed that if we represent a couple with a segment perpendicular to its plane, we can compound two couples by means of the parallelogram law.

In a successive work [Poinsot 1806], Poinsot demonstrated the existence of the central axis and gave vectorial proofs of the conservation of momentum and of moment of momentum in an isolated system. While in the first edition of this paper he did not say anything about the results obtained by Euler, in the subsequent editions, published as an appendix to the Eléments, Poinsot added an observation about the formula $G \cos \theta = L \cos \lambda + M \cos \mu + N \cos \nu$, which furnishes the value of the projection of the couple $G$ on the axis whose cosines are $\cos \lambda, \cos \mu, \cos \nu$ with respect to the coordinate axes:

“[This is] a very simple formula, which Euler gave in vol. VII of the New Proceedings of Petersburg, but to which he could arrive only by means of lengthy analytical calculation.” 20 [Poinsot 1803, 1842 ed., p. 355]

A different geometric representation of moments was developed by Poisson a little later [Poisson 1808]. Poisson remarked that the moment of a force about a point is numerically equal to the double of the area of a triangle having the vertex in the point and the force itself as its basis, and thus implicitly assumed that it can be represented geometrically by the triangle. Poisson was clearly inspired by Laplace’s theory of the invariable plane and by Poinsot’s couples. Euler’s formula is given in the form

$$D = A' \cos \epsilon' + A'' \cos \epsilon'' + A''' \cos \epsilon''',$$

19 “[Cette formule est] d’une simplicité et d’une élégance telle qu’on peut la regarder comme une des plus belles de la mécaniques.” (Author’s translation.)

20 “[elle est une] formule très-simple qu’Euler a donné dans le tome VII des Nouveaux Actes de Petersbourg, mais à laquelle il n’était parvenu que par de longs circuits d’analyse.” (Author’s translation.)
where $D$ is the plane area which represents the moment, $A'$, $A''$, $A'''$ are its projections on the coordinate planes and $\epsilon'$, $\epsilon''$, $\epsilon'''$ are the angles between $D$ and the coordinate planes.

Poisson included this theory in his *Traité de Mécanique*, one of the most influential mathematical textbooks of all time [Poisson 1811, vol. I, liv. I, ch. III], [Poisson 1833, vol. I, liv. III, ch. II]. Here, to distinguish between the two sides of a surface, he considered the directed straight line perpendicular to it; this is the first example of a surface oriented by means of a vector.\(^{21}\)

It should be noted that while in 1808 Poisson had given all the credit for the discovery of the formula for moments to Laplace (“these theorems on the invariable plane and on the composition of moments are due to M. Laplace”),\(^{22}\) in the second edition of the *Traité de Mécanique* [Poisson 1833, p. 544] these developments in the theory of moments were attributed to Euler alone (“these remarkable theorems are due to Euler”).\(^{23}\)

The new theory of moments was briefly taken up by Lagrange in the second edition of the *Mécanique analytique* [Lagrange 1811-15, vol. I, partie I, sect. III, §III, n. 16]. To demonstrate Euler’s formula, Lagrange used the vectorial representation of infinitesimal rotations. He started from the expression of the virtual work due to a small rotation of a rigid body,

$$L \, d\psi + M \, d\omega + N \, D\phi,$$

where $L$, $M$, $N$ are the moments of the force about the three axes of a rectangular Cartesian system of coordinates and $d\psi$, $d\omega$, $D\phi$ are the infinitesimal rotations about the same axes. Lagrange substituted the given rotations with their decomposition into three rotations about a second system of orthogonal axes, thus obtaining the moments about the new axes in the form

$$L \cos \lambda' + M \cos \mu' + N \cos \nu',$$

$$L \cos \lambda'' + M \cos \mu'' + N \cos \nu'',$$

$$L \cos \lambda''' + M \cos \mu''' + N \cos \nu''' ,$$

where $\lambda'$, $\mu'$, $\nu'$, $\lambda''$, $\mu''$, $\nu''$, $\lambda'''$, $\mu'''$, $\nu'''$ are the angles formed by the new axes with the original system. Lagrange remarked that this result had

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\(^{21}\) The earliest example of an oriented surface appeared just a few years before in L. Carnot’s *Géométrie de position*, where the two sides of a surface are described as painted in different colours [Carnot 1803, p. 94].

\(^{22}\) “Ces théorèmes sur le plan invariable et sur la composition des momens sont dus à M. Laplace.” (Author’s translation.)

\(^{23}\) “Ces théorèmes remarquables sont dus à Euler.” (Author’s translation.)
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been obtained by geometrical methods in the Novi commentarii for 1789, but Euler’s name was not mentioned.  

Still another geometric representation of moments was proposed by Binet in 1815 [Binet 1815]. While considering the motion of a rigid body with a fixed point $O$, he substituted every applied force $F$ with a force whose line of action is situated at a unitary distance from $O$ and whose moment about $O$ is the same as that of $F$, and said that this new force represents the moment of $F$ about $O$. Euler’s memoir is referred to in §III, which contains an analytical rephrasing of some portions of Poinsot’s theory of couples. Here Binet observes that the expression for the least total couple could also be obtained by means of the formula for the distance between two straight lines found by Euler.

In a second paper on the theory of moments, Binet introduced the vector representation of the areal velocity, for which he openly acknowledged the influence of Euler and Poinsot:

“The areal velocities can be composed following rules analogous to those for the composition and resolution of linear motions. It is not necessary for me to insist on this point, that the theorems of Euler and the research on moments of M. Poinsot have established without doubt, for our areal velocities are exactly the moments of ordinary velocities.” [Binet 1823, p. 164 (author’s translation)].

Some additional contributions to Euler’s formula were made by Antonio Bordoni, who expressed the formula in various forms and used it to solve several problems [Bordoni 1822]. The greatest part of his paper is dedicated to the resolution of different forms of the following problem: Given four concurrent straight lines in space and the moments of a system of forces about three of them, to find the moment about the fourth line. Thus, in effect, Bordoni was studying the generalization of Euler’s formula to non-orthogonal Cartesian axes.

After 1820 the time was ripe for someone to organize all the different views involved in the theory of moments into a unified formulation. It fell to

24 “Experience with [Lagrange’s book] has led me to the following working hypothesis:
1. There was little new in the Méchanique Analytique; its content derives from earlier papers of Lagrange himself or from works of Euler and other predecessors.
2. General principles or concepts of mechanics are misunderstood or neglected by Lagrange.
3. Lagrange’s histories usually give the right references but misrepresent or slight the content.”[Truesdell 1964, 1968 reprint, p. 246]

25 “Les vitesse aréolaires se combinent entre elles, d’après des règles analogues à celles de la composition et de la décomposition des mouvements linéaires; je n’ai pas d’isister sur cet objet, que les théorèmes d’Euler et les recherches de M. Poinsot sur les moments ont mis hors de doute, puisque nos vitesse aréolaires sont précisément les moments des vitesses ordinaires.”
Cauchy to do this, as he had done with many other branches of mathematics. In 1826 he published in vol. I of his *Exercices de Mathématiques* five papers in which he brought the theory to its final formulation [Cauchy1826a-1826f]. Except for the lack of a proper vectorial notation, his treatment is essentially modern. Cauchy’s moments are vectors, like Poinset’s couples and Binet’s *momens*, that represent Poisson’s surfaces.

The almost simultaneous appearance of several different theories of moments obviously led to some controversies over priority, which allow us to see how these mathematicians viewed their own work. The first controversy arose in 1827 between Cauchy and Poinsot. After the publication of Cauchy’s theory of moments, Poinsot accused Cauchy of having published results which were merely repetitions of his theorems on couples disguised under a different notation [Poisson 1827a]. Cauchy replied that his theory was more general, for it could be applied to every kind of physical entity which can be represented by a directed line segment [Cauchy 1827].

A second controversy began when Poisson published a short account of the recent history of the theory of moments, in which he asserted that Euler was the discoverer the vectorial composition of moments and maintained that Poinsot’s work was entirely derived from that of his predecessors [Poisson 1827]. Poinsot answered with a long and detailed article in which he observed that his theory of couples had introduced a *geometrical composition* of moments, whereas up to then there had been only the algebraic sum of certain expressions [Poisson 1827b].

This was the end of the polemics. Euler’s papers were then cited in Möbius’ *Lehrbuch der Statik* [Möbius 1837, §89-91], but not in the relevant portion of Grassmann’s first *Ausdehnungslehre* [Grassman 1844, §59]. Thereafter, they disappeared from the literature on the vectorial theory of moments.

As we have seen, Euler’s first memoir includes the formula for the distance between two straight lines. This result can also be found in Monge’s *Feuilles d’Analyse* [Monge 1801, §12-13] and in Cauchy’s *Leçons sur les Applications du Calcul infinitésimal a la Géométrie* [Cauchy 1826f, Prélém., Prob. VII], but their proofs are completely different from Euler’s, thus giving evidence that they had been found independently.

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Euler’s Contribution to Differential Geometry and its Reception

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1. Leonhard Euler’s various contributions to differential geometry

There was almost no mathematical discipline in the eighteenth century to which Euler did not contribute. Many of Euler’s contributions to special fields were appreciated and respected. But the case of differential geometry is different. Though Euler wrote articles on curve and surface theory throughout his life, there is almost no secondary literature concerning this particular aspect of his work. Euler himself mentions only a few of his own articles in differential geometry in any of his others. Only Dirk Struik dedicated a chapter to Euler in his “Outline of a history of differential geometry.” [Struik 1933]

The term “differential geometry” was first used by Luigi Bianchi (1856-1928) in an Italian textbook Lezioni di geometria differenziale. (Pisa 1886) In Euler’s time we take it to mean the theory of curves and surfaces. The theory of curves began with the rise of calculus and important results came quickly. Isaac Newton (1643-1727) determined an expression for the curvature of plane curves by means of his kind of calculus [Stiegler 1968]. Also Jakob Bernoulli (1654-1705) wrote papers about cycloids, catenary curves,
helical curves, spirals, circle of curvature, caustics, the elastic curve and its radius of curvature and the lemniscate [Weil 1999]. Curve theory quickly became a particularly well developed field. At first Euler too was interested in curve theory, but he soon achieved results in surface theory also. Euler was the first mathematician who worked successfully on surface theory.

In the following, only Euler’s main results shall be discussed. It is not possible to give a complete survey. The main contributions, however, will be mentioned. The order will be chronological.

1.1. First example of a minimal surface: the catenoid, 1744

In 1741 Euler moved from St. Petersburg to Berlin. That same year he became member of the Berlin academy, the Brandenburgische Sozietät der Wissenschaften, founded in 1700. In 1742 Euler also became an honorary member of the Academy of St. Petersburg. For the years 1744 to 1766 Euler was director of the mathematical class of the Academy in Berlin. After the position had been vacant for five years Pierre-Louis Moreau de Maupertuis (1698-1759) was appointed president of the academy in 1746. When Maupertuis died, there was another interregnum from 1759 to 1764. In 1764, however, Frederic II (1712-1786, reg. 1740-1786) named himself head of the academy. After he was not appointed president of the academy, in 1766 Euler left Berlin and returned to St. Petersburg.

In a letter to Maupertuis on March 14, 1746, Euler mentioned that he had started his work on Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, “A method for finding curved lines enjoying the properties of maximum or minimum, or solution of isoperimetric problems in the broadest accepted sense” [E65] when he was still in St. Petersburg. Once in Berlin, Euler gave his manuscript to his publisher Bousquet and it was published in September 1744 in Lausanne and Geneva. It contained 6 chapters [Fraser 2005]. With this work Euler founded a new discipline within analysis, the calculus of variations. Before Euler there were several individual problems, but after Euler there was a general calculus of variations.

There are many connections between the calculus of variations and differential geometry, for example geodesic lines. These can be treated by means of variations and they play an important role in differential geometry. In Chap. IV, §11 for example Euler is concerned with the shortest lines on a general curved surface.

\[1\] He wrote, “Cela s’entend de mon ouvrage même, que j’avois déjà achevé à Petersbourg.” Opera omnia (4) 6, p.60.

\[2\] Opera omnia (1) 24, p. XI.
In the next chapter, however, Euler asked: “To find the curve among all others of the same length which, if rotated around the axis $AZ$, delivers a solid, the surface of which is either a maximum or a minimum.” $^3$ The answer is the general equation of the catenaria, the chain line:

$$dx = \frac{cdy}{\sqrt{(b + y)^2 - cc}}.$$ 

This most outstanding result shows that the catenoid is a minimal surface. This was the first example of a minimal surface in history [E65, Chap.V, §47; p. 186f].

1.2. Definition of the curvature of a surface, 1767

Four years later Euler published his very famous textbook *Introductio in analysin infinitorum*, [E101-102], which soon became classic. The second volume contains the theory of curves and in an appendix surface theory, though it has only little on differential geometry, for example some remarks on singular points and asymptotes of plane curves, some osculation properties, and notes on concavity and convexity in relation to the sign of the radius of curvature [Struik 1933, p.102].

Indeed, Euler did not give a general theory of surfaces. He just treated several topics on solid surfaces: the intersection of a surface with a plane, especially sections of cylinders, cones and spheres, some second-order surfaces and the intersection of two surfaces. This last chapter included the theory of space curves [Reich 2005b].

But fifteen years later Euler had achieved spectacular results on surface theory. For the first time it was possible to give a definition of the curvature of a surface. On September 8, 1763 Euler presented to the Berlin academy his “Recherches sur la courbure des surfaces.” The paper was published in 1767 [E333].

In 1766 Euler left Berlin and returned to St. Petersburg to accept the invitation of Catherine II, who truly appreciated him as a first class scientist, a recognition which Frederic II in Berlin had never granted.

Euler began his “Recherches” by formulating the problem: “I will begin by determining the radius of curvature for a section of an arbitrary plane cutting the surface.” $^4$ He set out a three-step plan to achieve this goal:

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$^3$ Invenire curvam, quae inter omnes alias eiusdem longitudinis circa axem $AZ$ rotata producet solidum, cuius superificies sit vel maxima vel minima. [E65, Chap. V, §47]

$^4$ “Je commencerai par déterminer le rayon osculateur pour une section quelconque plane, dont on coupe la surface.”
“(i) If a surface, the nature of which is known, is cut by an arbitrary plane, to determine the curvature of this section.

(ii) If the plane of the section is perpendicular to the surface in the point Z, to determine the radius of curvature of this section in the same point Z.

(iii) An arbitrary surface being given, to find the osculating radius of a section EPZ, which forms an angle \( \varphi \) with the principal section.”

The result was a very long expression, which was transformed into a much easier one by means of:

- \( f \), the largest radius of curvature, which belonged to the section \( EF \), and
- \( g \), the smallest radius of curvature, which belonged to the section normal to the previous one. After a long calculation the result was:

\[
 r = \frac{2fg}{f + g - (f - g) \cos 2\varphi}
\]

where \( f \) and \( g \) are the radii of curvature of the principal sections and \( \varphi \) is the angle between an arbitrary normal section and the principal section.

This was an astonishing result indeed, which Euler expressed in his own words: “And so the measurement of the curvature of surfaces, however complicated, which appeared at the beginning, is reduced at each point to the knowledge of two radii of curvature, one the largest and the other the smallest, at that point; these two things entirely determine the nature of the curvature and we can determine the curvatures of all possible sections perpendicular at the given point.”

1.3. Developable surfaces and the so-called Gaussian variables, 1772

In March 1770 Euler presented to the academy in St. Petersburg his paper “About solids, the surfaces of which can be developed on the plane” (De solidis quorum superficiem in planum explicare licet, [E419]). In this paper, Euler extended his study of surfaces to developable surfaces, a totally new concept. For the first time he represented a point \( x, y, z \) on a surface as a function of two variables \( t \) and \( u \). These were later called Gaussian variables.

\[5\] “Ainsi le jugement sur la courbure des surfaces, quelque compliqué qu’il ait paru au commencement, se réduit pour chaque élément à la connaissance de deux rayons osculateurs, dont l’un est le plus grand et l’autre le plus petit dans cet élément; ces deux choses déterminent entièrement la nature de la courbure en nous découvrant la courbure de toutes les sections possibles, qui sont perpendiculaires sur l’élément proposé.” [E333, p. 22]
Euler began with the remark that in elementary geometry it is well known that cylinders and cones have the property that they can be flattened out, or “developed” into a plane, while, for example, the sphere does not have this property. He asked which other kinds of surfaces have the property that they are developable into a plane; this question is, according to Euler, a most notable, characteristic one.\(^6\)

Euler investigated the conditions on \(x\), \(y\), and \(z\), the coordinates of a point, and the two variables \(t\) and \(u\) describing the surface, i.e.

\[
dx^2 + dy^2 + dz^2 = dt^2 + du^2.
\]

The geometrical problem is therefore reduced to the solution of the following analytical problem: given the two variables \(t\) and \(u\) you have to find the six equivalent functions \(l\), \(m\), \(n\), \(\lambda\), \(\mu\) and \(\nu\), so that the formulas

\[
ldt + \lambda du, \ mdt + \mu du, \ and \ ndt + \nu du
\]

are integrable and further satisfy:

\[
\lambda\lambda + \mu\mu + \nu\nu = 1, \ l\lambda + m\mu + n\nu = 0
\]

The exposition was threefold:

(i) a solution by means of analytical principles,

(ii) a solution by means of geometrical principles, and

(iii) the application of the second to the first solution.

As a result Euler was able to prove that the line element of the surface has to be the same as the line element of the plane or, as he expressed it, “All surfaces which can be developed on a plane by means of flexibility and without stretching can be represented by the tangents of a spatial curve.”

According to Andreas Speiser, this paper of Euler has to be regarded as one of his very best mathematical achievements.\(^7\)

Euler also mentioned his success in a letter to Lagrange dated January 16/27, 1770: \(^8\)

“I have found a complete solution to the following problem: It is a matter of finding three functions, \(X\), \(Y\), \(Z\) of two variables \(t\) and \(u\) such that setting \(dX = Pdt + pdu\), \(dY = Qdt + qdu\), \(dZ = Rdt + rdu\), they will satisfy the following conditions:

\(^6\) “Quaesitio igitur hinc nascitur maxime notatu digna, quo charactere ea solida instructa esse oportet, quorum superficiem in planum explicare licet.”

\(^7\) *Opera omnia* (1) 28, p. XXIV: “so werden wir diese Arbeit als eine mathematische Höchstleistung bezeichnen dürfen.”

\(^8\) “...j’ai trouvé une solution complète du problème suivant: Il s’agit de trouver trois fonctions \(X\), \(Y\), \(Z\) de deux variables \(t\) et \(u\), telles que, posant \(dX = Pdt + pdu\), \(dY = Qdt + qdu\), \(dZ = Rdt + rdu\), on satisfasse aux conditions suivantes:
I. \( P^2 + Q^2 + R^2 = 1, \)
II. \( p^2 + q^2 + r^2 = 1, \)
III. \( Pp + Qq + Rr = 0. \)

Now, the nature of differentials requires the following additional conditions:

I. \( \frac{\partial P}{\partial u} = \frac{\partial p}{\partial t} \)
II. \( \frac{\partial Q}{\partial q} = \frac{\partial q}{\partial t} \)
III. \( \frac{\partial R}{\partial u} = \frac{\partial r}{\partial t}. \)

As an altogether singular thought led me to the solution of this problem, which I would have believed before would be impossible, I think that this discovery could become very important in the new part of integral calculus for which Geometry is indebted to you."

Euler should agree with Lagrange’s assessment of the importance of this result.

1.4. Orbiforms, 1781

Four years later, on May 12, 1774, Euler presented the Academy of St. Petersburg with a paper on orbiforms, which was published in 1781 with the
The problem itself had its origins in optics. Orbiforms are curves of constant breadth so named because the circle shares this property. Euler showed that the circle is not the only shape with this property. At first Euler investigated triangular curves, closed curves with three cusps that look like astroids. (See Fig 1.) Euler showed that the evolvents of these curves are curves of constant breadth.

Homer White gives a further discussion of this article, E513, in “The Geometry of Leonhard Euler” elsewhere in this volume.
1.5. Moving trihedral, spherical image, first Frenet formula, 1786

For a long time, Euler was not interested in the theory of space curves. But on May 28, 1775, he presented to the St. Petersburg Academy his paper “Easy method to investigate all points of intersection of curves which do not lie in the same plane” (Methodus facilis omnia symptomata linearum curvarum non in eodem plano sitarum investigandi, [E602]).

Alexis Clairaut (1713-1765) had been the first to treat space curves systematically. He published his results in Paris in 1731 under the title Recherches sur les courbes à double courbure. To investigate space curves analytically Clairaut used projections of space curves onto the planes of the coordinates. Euler, however, chose the arc length $s$ as the variable of a space curve, which made the presentation much more elegant. In §5 he introduced a unit sphere with its center at a point moving along the curve. This is equivalent to the introduction of the spherical image, which was later used by Gauß indexGauss, Karl Friedrich and now known as the Gauss map. Euler then defined the moving trihedral, i.e. the tangent, the normal and the second normal (binormal) and calculated their cosines (§17 and 18) as well as the radius of curvature (§10). In the second of the paper part he continued his calculations. The main results were, written in modern terminology:

$$\vec{t}' = \vec{h} \times \vec{b}, \quad \vec{h} = \vec{b} \times \vec{t}, \quad \text{and} \quad R \frac{dt}{ds} = \vec{h}. $$

This is the first of the three Frenet formulas.

1.6. Developable surfaces, rigidity of closed surfaces, 1862

This paper of Euler is a fragment, written by his students in a kind of notebook, the Adversarii mathematici. Euler again treated the development of surfaces. The problem is “To find two surfaces one of which may be transformed into the other so that in both of them homologous [corresponding] points keep the same distances from each other.”\footnote{“Invenire duas superficies, quarum alteram in alteram transformare liceat, ita ut in utraque singula puncta homologa easdem inter se teneant distantias.”} Euler proves that surfaces have this property if they have the same line elements.

The paper finishes with the following annotation:

“It is appropriate here to note one may not assume another surface other than the given one. In any case it is not clear how the functions $p, m,$ and $n$ have to be taken so that the surface has the given shape, for example a sphere. In both formulas the, two variables $r$ and $s$ can be
augmented to infinity and that this extension cannot be removed by any imaginary thing. Hence neither the sphere nor any other figure in a finite space can be described by these formulas. But as to the terminated or everywhere closed figures it looks as if they have to be judged in another way, because as soon as a solid figure is everywhere completed, it does not permit any further mutation. This can be understood by looking at these known figures that usually are called regular. Thus insofar as the spherical surface is complete, it dies not admit any mutation. Hence it is clear that such figures can be mutated insofar as they are not integer or everywhere closed. Yet, it is clear that the figure of the hemisphere is certainly mutable. But which kinds of mutations are possible seems to be a very difficult problem.”

2. Reception

The first reactions to Euler’s discoveries came from Italy. Lagrange and Euler had corresponded since 1754. There still exist 36 letters exchanged between the two through 1775, but Euler and Lagrange never met each other.

2.1. Joseph Louis Lagrange (1736-1813)

Euler’s and Lagrange’s common interests included the calculus of variations. Lagrange was 29 years younger than Euler and 10 years older than Monge. He was born in Turin and began his career at first in his home-town where, in 1755, he became professor of mathematics at the Royal Artillery School. In 1756 Lagrange became corresponding member at the academy in Berlin and ten years later he succeeded Euler in Berlin as director of the mathematical class of the Academy. In 1772 Lagrange was elected associé étranger at the Académie des sciences in Paris. This was reconfirmed when the Académie was reorganized in 1785. After Frederic II died, Lagrange left Berlin in 1787 and returned to Paris, where he immediately became pensionnaire vétéran at the academy and professor at the École Normale and at the École Polytechnique. One year later, in 1788, he was promoted to directeur of the academy. In 1795 Lagrange became membre résident de la section mathématiques and was elected président du bureau provisoire. When the Bureau des longitudes was founded in Paris in 1795,

\[\text{LOL-Ch24-P9 of 24}\]
Lagrange was among the founding members. In 1801 also the Sozietät der Wissenschaften in Göttingen chose Lagrange as a corresponding member.

At first it had been Euler’s “A method for finding curved lines enjoying the properties of maximum or minimum” [E65] which fascinated Lagrange. This led to a correspondence between Euler and Lagrange. In 1760/1 Lagrange published some of his own results of the discussion with Euler: “Essai d’une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies” [Lagrange 1760/1]. This work, like Euler’s book, is mainly devoted to the calculus of variations, but Langrange’s work also included as “Appendix I” a chapter on minimal surfaces: “Par la méthode qui vient d’être expliquée on peut aussi chercher les maxima et les minima des surfaces courbes, d’une manière plus générale qu’on ne l’a fait jusqu’ici.”

Lagrange derived the partial differential equation of minimal surfaces:

\[
\left( \frac{dP}{dx} \right) + \left( \frac{dQ}{dy} \right) = 0,
\]

\[
P = \frac{p}{\sqrt{1 + p^2 + q^2}}, \quad Q = \frac{q}{\sqrt{1 + p^2 + q^2}},
\]

where \( x, y \) and \( z \) are rectangular coordinates.

Later these equations were transformed into the more modern form [Reich 1973, p.313f]:

\[
r \left( 1 - q^2 \right) - 2ps + (1 + p^2) t = 0,
\]

\[
p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},
\]

\[
r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.
\]

Lagrange, however, did not mention the catenoid, which, as we described above, had been found by Euler.

Lagrange, though, did refer extensively to Euler’s theory of curved surfaces [E333]. Lagrange’s textbook *Théorie des fonctions analytiques* which was first published in 1797 and had a second edition in 1813, included a chapter devoted to “Des sphères osculatrices. Des lignes de plus grande et de moindre courbure. Propriétés de ces lignes” [Lagrange 1813, Chapter IX]. Lagrange emphasized that the results of Euler and Monge should be

\[12\] “By the method which comes to be explained, one can also find the maxima and minima of curved surfaces, in a more general manner than has been used before,” *Oeuvres de Lagrange*, vol.1, p.353-357.
appreciated by all geometers: “These properties of surfaces are very curious and they merit the full attention of geometers; they will have especially important applications for the arts,” \(^{13}\) and quoted [E333], [Monge 1780] and [Monge 1785].

2.2. Gaspard Monge (1746-1818)

Euler’s achievements in differential geometry were also influential among Monge and his school of students from Mézière as well as students from the École Polytechnique in Paris.

Gaspard Monge was 39 years younger than Euler. Born in Beaune, he began his career as a student of the École Royale Du Génie de Mézières. This school, founded in 1748, had as its aim the education of engineers. It had its best time during the years 1765-1775. Taton called these years “la grande periode” [Taton 1964, p.586-596]. In 1769 Monge became répétiteur de mathématiques (tutor of mathematics), at the age of 24, and in 1770 he became responsible for all mathematical and physical lectures at the École in Mézières.

In 1772 Monge was elected to be a corresponding member of the Parisian Academy, and in 1780 he became adjoint géomètre, replacing Vandermonde. In 1785 Monge was promoted to associé of the physics class, and in 1795 he was nominated membre résident. Also that year Monge started teaching at the newly founded École polytechnique after the school in Mézières had closed in 1794.

Monge’s main interest was geometry; descriptive geometry, design, analytical geometry and so on. And of course, he always was keenly aware of relationships of theory with practice and the applications of geometry especially to technology.

In 1771 Monge presented the Parisian Academy with his first paper on developable surfaces, “Mémoire sur les développées, les rayons de courbure et les différents genres d’inflexions des courbes à double courbure.” [Monge 1785] This paper had a new style, and Euler played no role in it. Monge proved several interesting theorems about space curves.

When Monge read Euler’s paper “About solids, the surfaces of which can be developed on the plane” [E419], Monge got even more interested in developable surfaces. He wrote a second paper, which he presented to the Academy in 1775 and published even earlier than the first one, “Mémoire sur les propriétés de plusieurs genres de surfaces courbes, particulièrement

\(^{13}\)“Ces propriétés des surfaces sont très-curieuses et méritent toute l’attention des géomètres; elles donnent lieu surtout à des applications importantes pour les arts” Oeuvres 9, p. 273.
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sur celles des surfaces développables, avec une application à la théorie des ombres et des pénombres” [Monge 1780]. Monge mentioned Euler:

Having started this material, on the occasion of a memoir of Mr. Euler in the 1771 volume\(^{14}\) of the Academy of St. Petersburg on developable surfaces and in which that illustrious Geometer gave the formulas for recognizing whether or not a given curved surface has the property of being able to be mapped to a plane, I arrived at some results which seem much simpler to me and easier to use for the same purpose.”\(^{15}\)

Indeed, Monge gave the following definition: “A surface is developable whenever, by supposing it to be flexible and inextensible, one may conceive of mapping it onto a plane, like cones and cylinders can be, so that the way in which it rests on the plane is without duplication or disruption of continuity.”\(^{16}\)

It is remarkable that Monge also characterized the developable surfaces with the terms “flexible et inextensible.” Monge managed to deduce the general differential equation of developable surfaces [Monge 1780, p.398]. As Taton had pointed out, there was a gap between Euler and Monge as far as styles were concerned: “Euler, of a profoundly analytic spirit, and Monge, dominated constantly by a sharp sense of geometric reality.”\(^{17}\)

2.3. Monge’s school

2.3.1. Graduate students from the École Royale du Génie de Mézières

The École Royal du Génie de Mézières was supposed to educate practitioners, so most of the students had no scientific ambitions. Nevertheless, two of Monge’s pupils at Mézières should be mentioned.

Charles Tinseau (1749-1822)

Tinseau began his studies in 1769 and finished as a military engineer in 1771. In 1773 Tinseau became correspondant at the Académie des sciences,

\(^{14}\)Here, Monge made an error. The correct date is 1772.
\(^{15}\)“Ayant repris cette matière, à l’occasion d’un Mémoire que M. Euler a donné dans le Volume de 1771, de l’Académie de Pétersbourg, sur les surfaces développables, et dans lequel cet illustre Géomètre donne des formules pour reconnaître si une surfache courbe proposée, jouit ou non de la propriété de pouvoir être appliquée sur un plan, je suis parvenu à des résultats qui me semblent beaucoup plus simples, et d’un usage bien plus facile pour le même objet.”
\(^{16}\)“Une surface est développable, lorsqu’en la supposant flexible et inextensible, on peut concevoir appliquée sur un plan, comme celles des cones et des cylindres, de manière qu’elle le touche sans duplication ni solution de continuité...” [Monge 1780, p.383]
\(^{17}\)“Euler, d’esprit profondément analytique, et Monge, dominé constamment par un sens aigu de la réalité géométrique.” [Taton 1951, p.21]
and in 1774 he presented his paper “Solution de quelques problèmes relatifs à la théorie des surfaces courbes et des courbes à double courbure” [Tinseau 1780]. The paper shows that Tinseau was directly influenced by Monge and that he did not quote Euler. Tinseau solved 17 problems, concerning, among other things the equation of the osculating plane to a space curve, the surface of the tangents to a curve and the theorem that the orthogonal projection of a space curve onto a plane has a point of inflexion, if its plane is perpendicular to the osculating plane [Struik 1933, p. 108]. In a second paper Tinseau treated several problems of ruled surfaces [Taton 1951, p.233f]. Afterwards, Tinseau made a military career. He campaigned against the French revolution and was later exiled.

Jean Baptiste Meusnier (1754-1793)

Monge’s second pupil at Mézières was of much greater importance and was much more recognized. Jean Baptiste Meusnier studied at the École du Génie from 1774 to 1775. On February 14 and 21, 1776, Meusnier presented his first and only paper, “Mémoire sur la courbure des surfaces” to the Académie Royale des Sciences in Paris and in June 1776 he became correspondant of the Academy. In 1784 he became adjoint géomètre and in 1785 associé de la classe de géométrie.

Monge made notes about the circumstances under which his young student got involved in the curvature of surfaces. As soon as he had arrived in Mézières, Meusnier visited Monge and asked him for a special project, hoping to prove his skills to Monge. Monge further reported:

“To satisfy him, I talked to him about the theory of Euler on the radii of maximum and minimum curvature of curved surfaces; I showed him the principal result and proposed that he look for its proof. The next morning in my office, he gave me a short paper, containing his proof; but what was remarkable was that the reasoning it used was more direct, and the path he followed was much shorter than Euler’s had been. The elegance of this solution and the little time that it had cost to him gave me an idea of his sagacity and all the work that he has undertaken since have the same evidence of his exquisite sense of the nature of the things.”

18 “Pour le satisfaire, je l’entretins de la théorie d’Euler sur les rayons de courbure maxima et minima des surfaces courbes; je lui en exposai les principaux résultats et lui proposai d’en chercher la démonstration. Le lendemain matin, dans les salles, il me remit un petit papier, que contenait cette démonstration; mais ce qu’il y avait remarquable, c’est que les considérations qu’il avait employées étaient plus directes, et la marche qu’il avait suivie était beaucoup plus rapide que celles dont Euler avait fait usage. L’élégance de cette solution et le peu de temps qu’elle lui avait coûté me donnèrent une idée de sa sagacité et de ce sentiment exquis de la nature des choses dont il a donné des preuves mul-
Meusnier’s paper “Mémoire sur la courbure des surfaces,” presented in 1776, was his only mathematical paper and it was not published until nine years later [Meusnier 1785]. As the title suggests, Meusnier’s work was based in part on Euler’s paper of the same title. [E333] Meusnier wrote:

“Mr. Euler has treated the same material in a very beautiful Memoir published in 1760 by the Academy in Berlin. This famous Geometer considers the question in a way different from the one we have just described. He makes the curvature of a surface element depend on the various sections that one can make by cutting it with planes.”

Indeed, Meusnier used different methods than Euler and he managed to add some new results. The theorem of Meusnier is still presented in some modern textbooks and is mentioned in modern mathematical dictionaries:

The centre of curvature at a point $P$ for a curve on a surface is the projection upon its osculating plane of the centre of curvature of that normal section of the surface which is tangent to the curve $P$.

In his paper Meusnier solved the following five problems:

1. To determine the different positions that the tangent plane can have in the understanding of a surface element.
2. To determine the radius of curvature of the section made on a surface element by a plane in any given position.
3. To determine the kinds of surfaces for which the two radii of curvature are always equal.
4. Among all surfaces which can be made to pass through a given perimeter formed by a curve of double curvature, to find that for which the area is the least.
5. To find the general equation for developable surfaces.”

"M. Euler a traité la même matière dans un fort beau Mémoire, imprimé en 1760 parmi ceux de l’Académie de Berlin. Cet illustre Géomètre envisage la question d’une manière différente de celle que nous venons d’exposer; il fait dépendre la Courbure d’un élément de surface, de celle des différentes sections qu’on y peut faire en le coupant par des plans” [Meusnier 1785, p.478f].

"1. Déterminer les différentes positions que peut avoir le plan tangent dans l’étendue d’un élément de surface?
2. Déterminer le rayon de Courbure de la section faite dans un élément de surface par un plan quelconque donné de position.
3. Déterminer quelles sont les surfaces pour lesquelles les deux rayons de Courbure sont toujours égaux.
4. Entre toutes les surfaces qu’on peut faire passer par un périmètre donné, formé par une courbe à double Courbure, trouver celle dont l’aire est la moindre.
5. Trouver l’équation générale des surfaces développables.”